3 Stochastic Equilibrium Concepts

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- **3 stochastic equilibrium concepts (for non-stochastic environments):**
  - Sunspot Equilibrium. (Cass and Shell)
  - Lottery Equilibrium. (Prescott and Townsend)
  - Correlated Equilibrium. (Aumann)
- Different histories
- Different motivations
- Are they related?
- How are they related?
- 2 different uses of "equilibrium"
  - market clearance
  - co-ordinated beliefs
- 2 axioms of Rational Expectations
  - smart agents
  - co-ordinated beliefs

Uncertainty

- *Extrinsic uncertainty* does not affect the fundamentals (e.g. preferences, endowments, technologies, payoffs), but might affect outcomes (prices, allocations, etc.).
  - Basis for NM axioms
  - "Excess volatility" in macro that can worsen matters
  - Random strategies in game theory that can improve matters
  - "Convexification" of economy that can improve matters
- *Intrinsic uncertainty* does affect fundamentals. (Arrow 1953, 1964)
- Extrinsic uncertainty can be thought of as uncertainty generated by the economy.
- Intrinsic uncertainty is transmitted through the economy.
- Extrinsic uncertainty can be seen as limiting case of intrinsic uncertainty.

Sunspot Equilibrium

- Macro
- Volatility
  - Excess volatility
  - Shiller
- What is "excess volatility"?
- Well-known that expectations can lead to economic volatility and business cycles
- Can rational expectations support excess volatility?
- (Unfair) spoof on W. S. Jevons. Sir Arthur Shuster.
- Typically **not** mere randomization over certainty equilibria!
- Does RE inevitably lead to good outcome?

- Simple Certainty Exchange Economy

$$y_h \in \mathrm{R}^l_{++}$$
$$e_h \in \mathrm{R}^l_{++}$$
$$u_h(\cdot) \text{ smooth and strictly concave}$$
$$h = 1, ..., H$$
$$q \in \mathrm{R}^l_{++}$$

- Consumer h

$$\max u_h(y_h)$$

subject to

$$q \cdot y_h = q \cdot e_h.$$

q is an equilibrium price vector if

$$\sum y_h = \sum e_h.$$

Introduce sunspots: $s = \alpha, \beta; \pi(\alpha) + \pi(\beta) = 1.$

- Build the sunspots economy from the certainty economy

$$\omega_h = (\omega_h(\alpha), \omega_h(\beta)) = (e_h, e_h) \in \mathbb{R}^{2l}_{++}$$

symmetry

$$\omega_h(\alpha) = \omega_h(\beta)$$

$$v_h(x_h(\alpha), x_h(\beta)) = \pi(\alpha) u_h(x_h(\alpha)) + \pi(\beta) u_h(x_h(\beta))$$

symmetry permuting

$$(\pi(\alpha); x_h(\alpha)) \text{ and } (\pi(\beta); x_h(\beta))$$

[More generally, Balasko (1983): $v_h(\sigma x_h; \sigma\pi) = v_h(x_h; \pi)$.]

$$p = (p(\alpha), p(\beta)) \in \mathbb{R}^{2l}_{++}$$

- $$\max v_h(x_h(\alpha), x_h(\beta)) = \pi(\alpha)u_h(x_h(\alpha)) + \pi(\beta)u_h(x_h(\beta))$$

  subject to

  $$\begin{aligned} p(\alpha)\cdot x_h(\alpha) + p(\beta)\cdot x_h(\beta) &= p(\alpha)\cdot\omega_h(\alpha) + p(\beta)\cdot\omega_h(\beta) \\ &= (p(\alpha)+p(\beta))\cdot e_h. \end{aligned}$$

  $(p(\alpha), p(\beta))$ is an equilibrium if

  $$\sum x_h(\alpha) = \sum e_h = \sum x_h(\beta).$$

- *Sunspots* are said to *matter* if for some h

  $$x_h(\alpha) \neq x_h(\beta)$$

  or, more generally, if

  $$u_h(x_h(\alpha)) \neq u_h(x_h(\beta)).$$

  Otherwise, sunspots do not matter.

- Sunspot equilibria are symmetry breaking: asymmetric solutions to symmetric equations.
- Until the "sunspots revolution", economists might have claimed that the RE economy is immune to sunspot effects. There is *some* basis for this conjecture. Consider the next two examples.

- **Cass-Shell Immunity Theorem:**
  In the finite strictly convex economy with complete markets and unrestricted participation with shared probability beliefs, sunspots do not matter.
- Consider a competitive equilibrium allocation x for the sunspots economy. It is PO *in the sunspots economy*. Assume that $x_h(\alpha) \neq x_h(\beta)$ for some h. Calculate the weighted averages,

$$\bar{x}_h = \pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta).$$

$$\begin{aligned} \sum x_h(s) &= \sum(\pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta)) \\ &= \pi(\alpha)\sum x_h(\alpha) + \pi(\beta)\sum x_h(\beta) \\ &\leq \pi(\alpha)\sum \omega_h(\alpha) + \pi(\beta)\sum \omega_h(\beta) \end{aligned}$$

for $s = \alpha, \beta$. So $x_h = (\bar{x}_h, \bar{x}_h)$ for all h is a feasible allocation.

$$
\begin{aligned}
v_h(\bar{x}_h(\alpha), \bar{x}_h(\beta)) &= \pi(\alpha)u_h(\bar{x}_h) + \pi(\beta)u_h(\bar{x}_h) \\
&= \pi(\alpha)u_h(\bar{x}_h) + (1 - \pi(\alpha))u_h(\bar{x}_h) \\
&= u_h(\bar{x}_h) \\
&= u_h(\pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta)) \\
&> \pi(\alpha)u_h(x_h(\alpha)) + \pi(\beta)u_h(x_h(\beta)) \\
&= v_h(x_h(\alpha), x_h(\beta))
\end{aligned}
$$

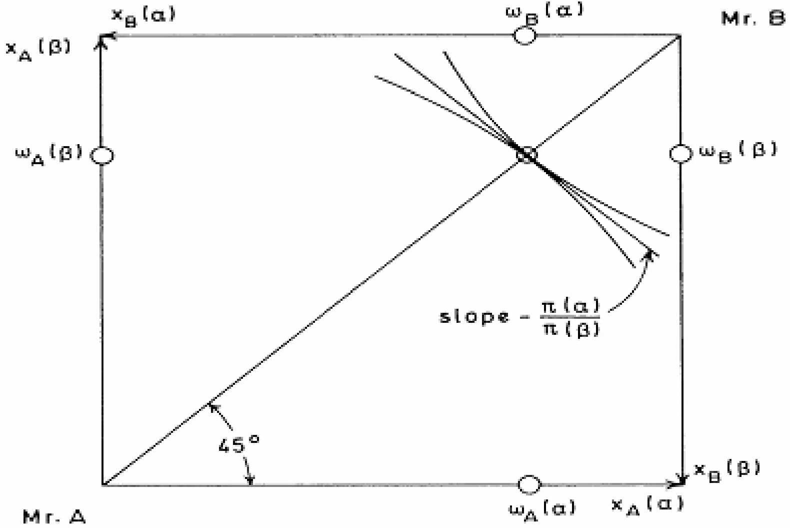
- Contradiction, hence we have

$$x_h(\alpha) = x_h(\beta) = y_h$$

$$p(\alpha)/\pi(\alpha) = p(\beta)/\pi(\beta) = q.$$

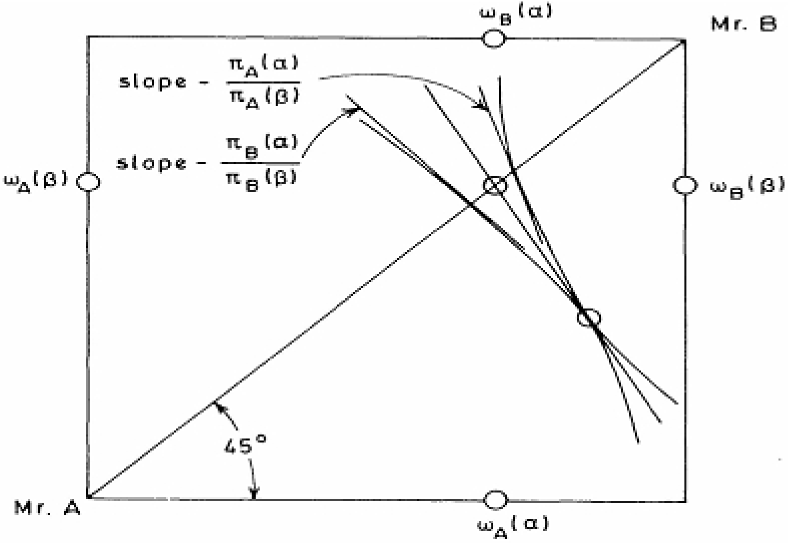
Therefore solution is symmetric and equivalent to that in certainty economy. Sunspots do not matter.

Example 1: Sunspots Do Not Matter



$\pi_h(\alpha) = \pi(\alpha)$ and $\omega_h(\alpha) = \omega_h(\beta)$ for $h = A, B$

Example 2: Sunspots Matter



$\pi_A(\alpha) \neq \pi_B(\alpha), \quad \omega_h(\alpha) = \omega_h(\beta) \quad \text{for} \quad h = A, B$

Other sources of SE (The Philadelphia Pholk "Theorem")

- Restricted market participation (naturally arising in OG economies)
- "Double infinity" of consumers and dated commodities (the infinite horizon in OG models)
- Incomplete markets
- Asymmetric information
- Imperfect competition (e.g., market games). If endowments are not PO, then there always exist proper SE. (Necessary and sufficient)
- Consumption and production externalities
- Monetary indeterminacy
- Nonconvexities. Here SE is PO in sunspots economy, while CE is PO in certainty economy.
- Search
- Bank runs

Example 3: Indivisible good

- $h = 1, 2$. $e_h = 1/2$. $X_h = \{0, 1\}$.
  $u_h(1) = b > u_h(0) = 0$.
- Good allocation would be $x_h = 1$ with probability $1/2$. Can this be supported as a SE? Yes.
- Let $s = \alpha, \beta$. $\pi(\alpha) = \pi(\beta) = 1/2$.

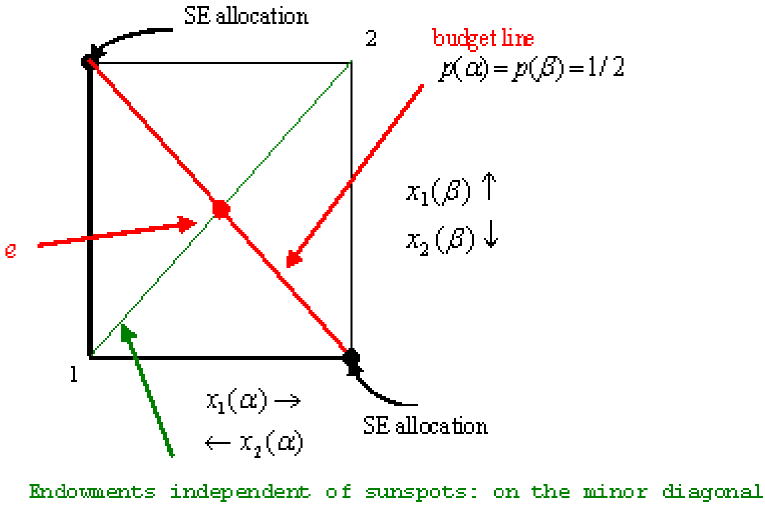
$$\max\{\frac{1}{2}u_h(x_h(\alpha)) + \frac{1}{2}u_h(x_h(\beta))\}$$

subject to

$$\begin{aligned} p(\alpha)x_h(\alpha) + p(\beta)x_h(\beta) &= p(\alpha)\omega_h(\alpha) + p(\beta)\omega_h(\beta) \\ &= (p(\alpha) + p(\beta))\frac{1}{2} \end{aligned}$$

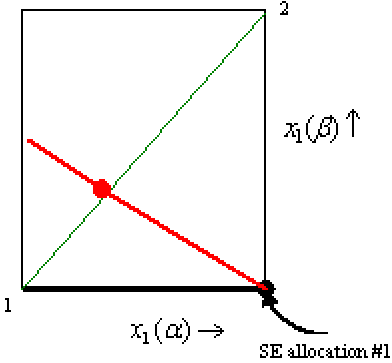
Normalize

$$p(\alpha) + p(\beta) = 1.$$



Edgeworth Square: $e_h = 1/2$ for $h = 1, 2$. $\pi(\alpha) = \pi(\beta) = 1/2$
Prices co-linear with probabilities and endowments

Let $\omega_1 = 1/3$, $\omega_2 = 2/3$.



Edgeworth Square: $0 \leq \pi(\alpha) \leq \frac{1}{2}$; $\frac{1}{2} \leq \pi(\beta) \leq 1$. $(p(\alpha), p(\beta)) = (\frac{1}{3}, \frac{2}{3})$.
Unless $(\pi(\alpha), \pi(\beta)) = (\frac{1}{3}, \frac{2}{3})$:

(1) SE allocation outside the core for sufficiently large replication of the economy.

(2) SE allocation not supported with continuous randomizing device.

Continuous random variable with pdf $\phi(s)$ defined on S.
Sunspot consumer problem:

$$\max \int_s u_h(x_h(s))\phi(s)ds$$

subject to

$$\int_s p(s)x_h(s)ds \leq \int_s p(s)e_h ds$$

and

$$x_h(s) \in \{0, 1\} \text{ for } s \in S.$$

Lottery Equilibrium

- In the sunspots economy, basic commodity is chocolate delivered in state s. In the lottery economy, basic commodity is chocolate delivered with probability π.
- Moral hazard constraint yields effective non-convex budget sets.
- Standard work-week (Rogerson)

Lottery Equilibrium, or "Buying Probability"

- The same indivisible good economy: $X_h = \{0, 1\}$.

$$\max_{\pi_h} \pi_h b$$

subject to

$$q\pi_h b \leq qe_h = (\frac{1}{2})q$$

Result:

$$\pi_h = \frac{1}{2},\ h = 1, 2$$

- PT Lottery requires continuum of agents, so that "law of large numbers" permits individuals to flip their own coins (but do not cheat).
- For finite economies (Garratt), correlated randomizing device can replace state-specific trading of SE.

- Prescott and Townsend: Continuum of agents
  Garratt: Extension to finite number of agents
- Example: X_h non-convex, think indivisible goods, n finite positive integer, sunspot and lottery variables are continuous.
- Consumer problem in sunspot economy:

$$\max_{x_h} \int_s u_h(x_h(s))\phi(s)ds$$

subject to

$$\int_s p(s) \cdot x_h(s)ds \leq \int_s p(s) \cdot e_h ds$$

$x_h(s) \in X_h$ for each s.

- Result: Every SE allocation can be supported by prices $p(s)$ which are colinear with $\phi(s)$, i.e. $(p(s)/\phi(s))$ is constant. (Garratt, Keister, Qin and Shell)

- Consumer problem in Garratt lottery economy with many indivisible goods:

$$\max_{\delta_h} \int_c u_h(c)\delta_h(dc)$$

subject to

$$q \cdot \int_c c\delta_h(dc) \leq q \cdot e_h,$$

where δ_h belongs to the set of probability measures over the (finite) set of commodity baskets C.

- Lottery Equilibrium: $q \in \mathbb{R}^l_+$ and allocation δ s.t.
  - (1) (q, δ) solves consumer's problem for each h;
  - (2) There is a joint lottery L s.t. δ_h is the marginal distribution for each h and there is coordination of supply and demand.

- Garratt, Keister, Qin and Shell:
  When the randomizing device is continuous, the set of SE allocations and the set of Garratt LE allocations are identical.
  - SE implements LE
  - LE reduced form (for computation) of SE
  - Correlation substitutes for contracts?
- Extension: same are true if there is a finite number of equiprobable states.
- Garratt, Keister and Shell:
  If there is a finite number of non-equiprobable states, the set of SE allocations and the set of LE allocations can differ.
- States of nature can possess "monopoly power".
  - $p^i(\frac{1}{4}) > \frac{1}{2}p^i(\frac{1}{2})$
  - $p^i(s \cup s') = p^i(s) + p^i(s')$, but
  - $\pi(s) = \pi(s')$ does not imply $p^i(s) = p^i(s')$.

Correlated Equilibrium

- Aumann (matrix games)

6, 6	2, 7
7, 2	0, 0

Chicken

NE payoffs: $(2,7), (7,2), (4\frac{2}{3}, 4\frac{2}{3})$

- Can we avoid a $(0,0)$ crash?

$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	0

How to compare SE and Corr E

SE:	**market**	economy
Corr E:	matrix	**game**

- Answer: Market Game (Shapley-Shubik)
  GE for Imperfect Competition

Certainty Market Game Γ

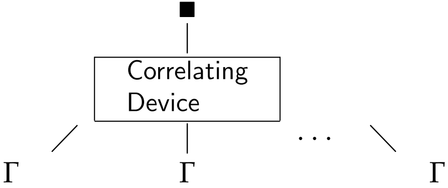
$$x_h^i = e_h^i - q_h^i + \frac{b_h^i}{B^i} Q^i$$

if

$$\sum_{i=1}^{l} (q_h^i / Q^i) B^i \geq \sum_{i=1}^{l} b_h^i$$

$$\max u_h(x_h)$$

- Peck and Shell; Peck, Shell and Spear:
  There are interior PSNE to Γ parameterized by q.
- Peck and Shell:
  There are proper Corr E to Γ if and only if ω is not PO.



- Corr E has correlating device outside rules. Complicated solution to simple game Γ.
- SE puts correlating device inside rules allowing through Arrow securities trade across states. Simple solution to complicated game.
- Hence, given a randomizing device: Every Corr E is a SE, but not the converse. Former are "self-enforcing". Latter are not.
- There is a proper SE to market game if and only if ω is not PO.

Summary

- SE first arose in OG model (Malinvaud lecture, *MD* forthcoming) to expand RE concept and question Lucas (1972, *JET*) conclusions. Further alternatives to RE?
- LE first arose in moral hazard model (Prescott and Townsend) to "convexify" economy and improve allocations.
- Corr E (Aumann) broaden the set of solutions in games with eye to achieving better allocations.
- SE and LE are formally very close (but not identical). LE is at its best when law of large numbers can be applied (continuum of agents). SE can decentralize LE that are based on Garratt correlating device.
- Corr E is complicated solution to simple game. SE is simple solution to complicated game. Corr E $\subset$ SE *for a given randomizing device*. Corr E does not allow transfers of resource across states.

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Appendix (Example 4)

- Pure exchange, taxes denominated in money.
- Boot-strapped construction of SE which is not mere randomization over CE.

  $l = 1,\ x_h \in \mathrm{R}_{++},\ \tau_h \in \mathrm{R},\ u_h = \log x_h,\ p^m \in \mathrm{R}_+.$
-

$$\begin{aligned} e &= (e_1, e_2, e_3) = (20, 10, 5) \\ \tau &= (\tau_1, \tau_2, \tau_3) = (5, 0, -5) \end{aligned}$$

- **CE:**

$$\begin{aligned} x_1 &= 20 - 5p^m = \tilde{e}_1 \\ x_2 &= 10 = \tilde{e}_2 \\ x_3 &= 5 + 5p^m = \tilde{e}_3 \end{aligned}$$

$$p^m \in [0, 4).$$

Sunspots:

- Mr.3 not able to participate in the securities market.
- Mr.1 and Mr.2 are unrestricted

$$s = \alpha, \beta \qquad \pi(\alpha) = 3/4 \qquad \pi(\beta) = 1/4$$

$$p^m(\alpha) = 1 \qquad p^m(\beta) = 2$$

- For $h = 3$

$$\begin{aligned} x_3(\alpha) &= 5 + 5 = 10 = \tilde{\omega}_h(\alpha) \\ x_3(\beta) &= 5 + 10 = 15 = \tilde{\omega}_h(\beta) \end{aligned}$$

For $h = 1, 2$

$$\max \frac{3}{4} \log x_h(\alpha) + \frac{1}{4} \log x_h(\beta)$$

s.t.

$$p(\alpha)x_h(\alpha) + p(\beta)x_h(\beta) = p(\alpha)\tilde{\omega}_h(\alpha) + p(\beta)\tilde{\omega}_h(\beta)$$

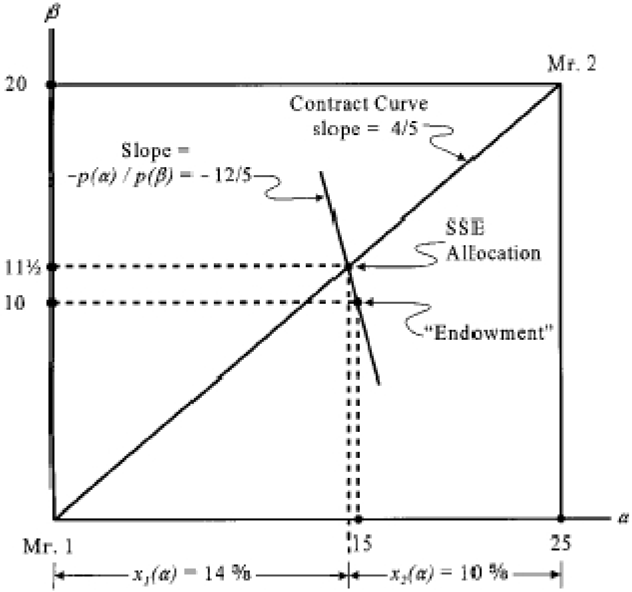
- $$\tilde{\omega}_1(\alpha) + \tilde{\omega}_2(\alpha) = 35 - 10 = 25$$

- $$\tilde{\omega}_1(\beta) + \tilde{\omega}_2(\beta) = 35 - 15 = 20$$

  Tax-adjusted Edgeworth box is 25×20, **not** square!

- NM utility + logs yields for market clearing:

$$\frac{p(\alpha)}{p(\beta)} = \frac{\pi(\alpha)}{\pi(\beta)} \left[\frac{\tilde{\omega}_1(\beta) + \tilde{\omega}_2(\beta)}{\tilde{\omega}_1(\alpha) + \tilde{\omega}_2(\alpha)} \right] = \frac{3/4}{1/4} \left[\frac{20}{25} \right] = \frac{12}{5}.$$


β
20
Mr. 2
Contract Curve
slope = 4/5
Slope =
−p(α) / p(β) = - 12/5
SSE
Allocation
11½
10
"Endowment"
α
Mr. 1
15
25
x₁(α) = 14 ⅝
x₂(α) = 10 ⅜