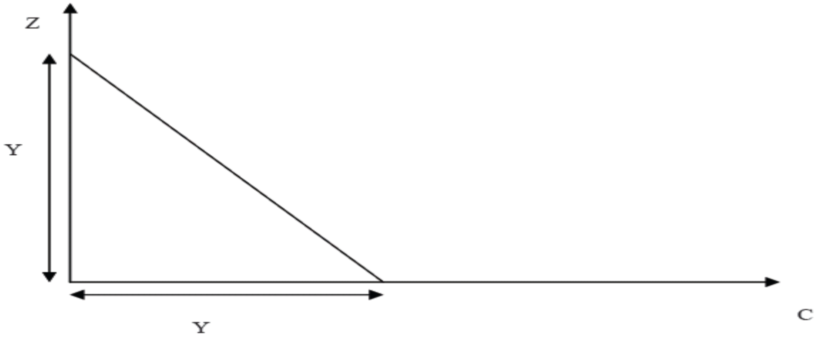
Production: Production Functions

- $Y = F(K, L)$
- Output Y is "cracked" or reformed into $C \geqslant 0$ and $Z \geqslant 0$ by $C + Z = Y$



- Slope of PPF= -1, Supply price of new machines in terms of consumption $\frac{P_Z}{P_C} = P_Y = 1$

- Constant-return-to-scale(CRS) means that F is homogeneous of degree one, so that $\theta Y = F(\theta K, \theta L)$ for $(Y, K, L) \geqslant 0$ and all scalers $\theta > 0$

- Example: log linear (Cobb-Douglas or Wicksell-Cobb): $Y = AK^a L^b$, where $A > 0$ is a scaler, $0 < a < 1, b = 1 - a$

- Let $\theta = 1/L$, then $\frac{Y}{L} = F(\frac{K}{L}, 1)$ or $y = f(k)$
  For Cobb-Douglas: $y = Ak^a$

- Example 2 (Fixed coefficient ot Leontief)
  $Y = A\min[aK, bL]$
  Full employment of both factors if and only if $aK = bL$,
  $\frac{Y}{L} = A\min[ak, b] = f(k)$
  Not differentiable, not smooth.

- For the smooth case, we have
  "Profits" $= F(K, L) - rK - wL$
  $\frac{\partial Y}{\partial K} = F_K = \frac{\partial F}{\partial K} = r$
  $\frac{\partial Y}{\partial L} = F_L = \frac{\partial F}{\partial L} = w$

- $Y = Lf(\frac{K}{L})$
  $\frac{\partial Y}{\partial K} = Lf'(\frac{K}{L})/L = f'(k) = r$
  $\frac{\partial Y}{\partial L} = f(\frac{K}{L}) + Lf'(\frac{K}{L})(-\frac{K}{L^2}) = f(k) - kf'(k) = w$

- Checking for adding up:
  $rk + w = kf'(k) + f(k) - kf'(k) = f(k) = y$

- Neoclassical Assumptions:
  Positive output: $f(k) > 0$ for $0 < k < \infty$
  Positive marginal product: $f'(k) > 0$ for $0 < k < \infty$
  Decreasing marginal product: $f''(k) < 0$ for $0 < k < \infty$
- Inada condition:
  $f(0) = 0 \quad f(\infty) = \infty$
  $f'(0) = \infty \quad f'(\infty) = 0$

Helps to insure uniqueness of steady state.

- Shows that Cobb- Dauglas satisfies the neo-classical conditions and the Inada conditions

- Neoclasical technology produces a one-to-one link between capital intensity k and the wage-interest ration $\omega = \frac{w}{r}$.Hence we can describe the technology by $\omega(k)$ or $k(\omega)$

Proof:

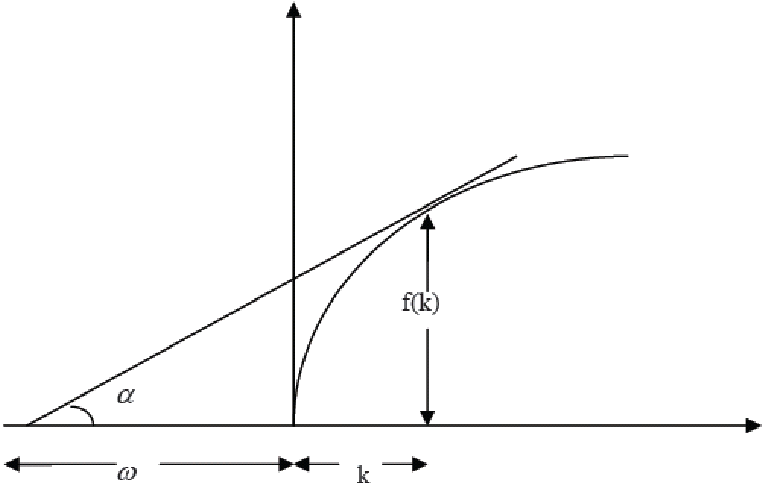
$$\omega = \frac{w}{r} = \frac{f(k)-kf'(k)}{f'(k)} = \frac{f(k)}{f'(k)} - k$$

$$\frac{d\omega}{dk} = \frac{f'(k)}{f'(k)} - \frac{f(k)f''(k)}{[f'(k)]^2} - 1 = -\frac{f(k)f''(k)}{[f'(k)]^2} > 0$$

$$\frac{dk}{d\omega} = -\frac{[f'(k)]^2}{f(k)f''(k)} > 0$$

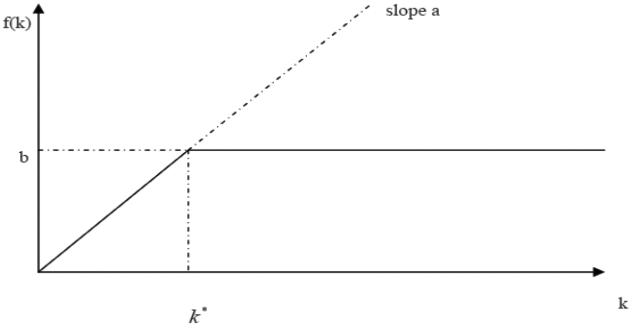
graphically proof, next slide

Production: Production Functions



- $\tan \alpha = f'(k) = \frac{f(k)}{k+x}$
  solve for x
  $k + x = \frac{f(k)}{f'(k)}$
  $x = \frac{f(k)}{f'(k)} - k$
  Hence $x = \omega$Hence $k(\omega)$ and $\omega(k)$ are increasing functions.
- *Probelm:*
  Do all of the previous exercises for Cobb-Douglas case:
  $Y = AK^a L^{1-a}$

- Leontief case. Caution: Use calculus at your own risk. (generalized sub-gradients)
  $y = \min(ak, b) = f(k)$



For $0 < k < k^*$, $r = a$, $w = 0$
For $k^* < k$, $r = 0$, $w = b$
For $k^* = k$, $rk + w = ak = ab$

- CES Production
  Referece: Arrow, Chenery,Minhas,&Solow The Review of Economics and Statistics, Vol. 43, No. 3 (Aug., 1961), pp. 225-250
  $\sigma = \frac{Ek}{E\omega} = \frac{\frac{dk}{d\omega}}{\frac{k}{\omega}} = \frac{-f'(f-kf')}{kff''} > 0$
- Cobb-Douglas case
  $f = Ak^a$
  $f' = Aak^{a-1}$
  $f'' = Aa(a-1)k^{a-2}$
  $\sigma = \frac{Ek}{E\omega} = \frac{\frac{dk}{d\omega}}{\frac{k}{\omega}} = \frac{-f'(f-kf')}{kff''} = \frac{-Aak^{a-1}(Ak^a-kAak^{a-1})}{kAk^aAa(a-1)k^{a-2}} = 1$
  $\sigma = 1$ *for* Cobb-Douglas

- Leontiff
  $\sigma = 0$
- Data:
  $0 << \sigma << 1$
- CES PF:
  $Y = (AK^{-\rho} + BL^{-\rho})^{-1/\rho}$
  $y = (Ak^{-\rho} + B)^{-1/\rho}$
  $\sigma = \frac{1}{1+\rho}$
- Show CES is neoclassical for y, A,and B greater than zero and ρ greater than minus one, but the Inada Condition will not hold

2-sector production:

- *Reference:* H. Uzawa, "On a Two-Sector Model of Economic Growth II," Review of Economic Studies, Vol June 1963, 30(2): 105-118.
- For 2, think C. For 1, think I or Z. $P = \frac{P_Z}{P_C}$
  - $Y_j = F_j(K_j, L_j) \quad j = 1, 2$
  - (FE) $K_1 + K_2 = K, \;\; L_1 + L_2 = L$
  - $Y = Y_2 + PY_1$
  - $k = \frac{K}{L}$, $y = \frac{Y}{L}$,
  - $k_j = \frac{K_j}{L_j}$, $y_j = \frac{Y_j}{L}$,
  - $l_j = \frac{L_J}{L}$

2-sector production:

Hence

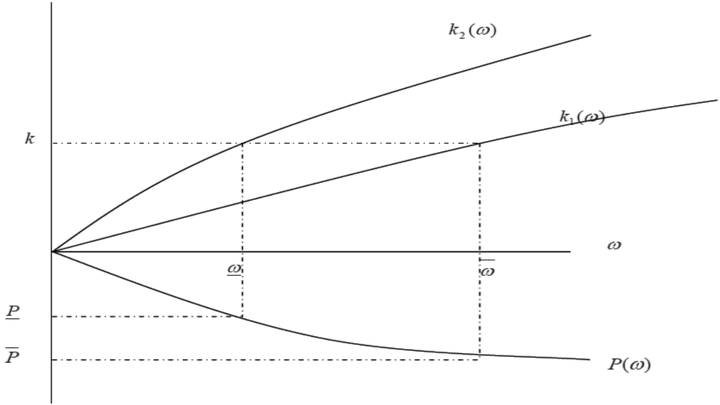
- $y_j = l_j f_j(k_j) \quad j = 1, 2$
- (FE) $k_1 l_1 + k_2 l_2 = k$
- $P = \frac{f_2'(k_2)}{f_1'(k_1)}$
- $\omega = \frac{f_j(k_J)}{f_J'(k_J)} - k_j > 0$
- $y_1 = (\frac{k_2 - k}{k_2 - k_1}) f_1(k_1)$
- $y_2 = (\frac{k - k_1}{k_2 - k_1}) f_2(k_2)$
- l_1 and l_2 are positive because k is a convex combination of $k_1$and k_2
- $\frac{dk_J}{d\omega} = \frac{-[f_j']^2}{f_j f_j''}$

$k_j(\omega)$ for $j = 1, 2$ describes the technology.
Exercises: Show the expressions for $k_j(\omega)$ if the 2 PF's are Cobb-Douglas.

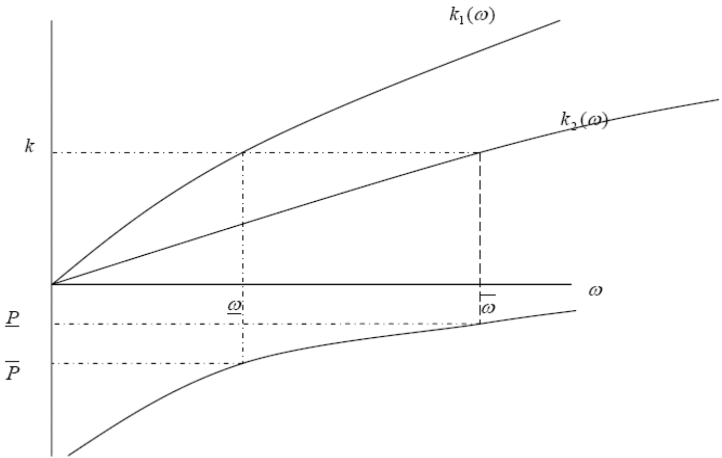
2-sector production:

- Log differentiating P w,r,t ω yields
  - $\frac{1}{P}\frac{dP}{d\omega} = \frac{1}{k_1(\omega)+\omega} - \frac{1}{k_2(\omega)+\omega}$
  - $\frac{dP}{d\omega} \begin{array}{c} > \\ = \\ < \end{array} 0 \text{ if } \begin{array}{c} k_2(\omega) > k_1(\omega) \\ k_2(\omega) = k_1(\omega) \\ k_2(\omega) < k_1(\omega) \end{array}$
  - $k_2(\omega) > k_1(\omega)$ thought to be "usual"

usual case $k_2(\omega) > k_1(\omega)$

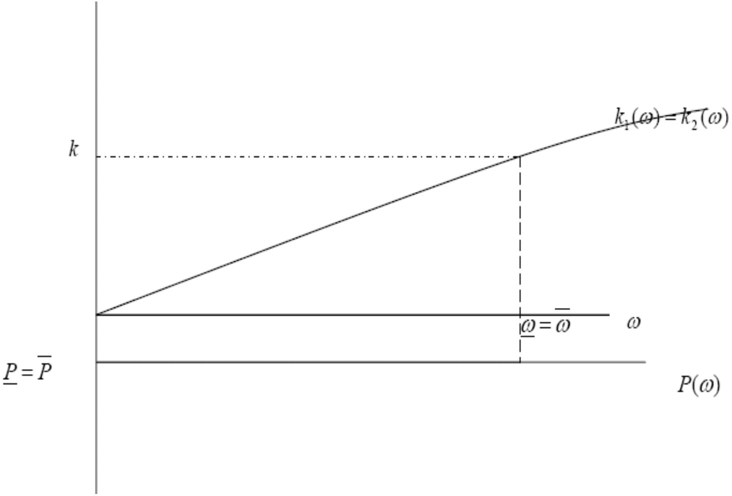


case $k_1(\omega) > k_2(\omega)$



2-sector production:Harrod-Johnson diagram

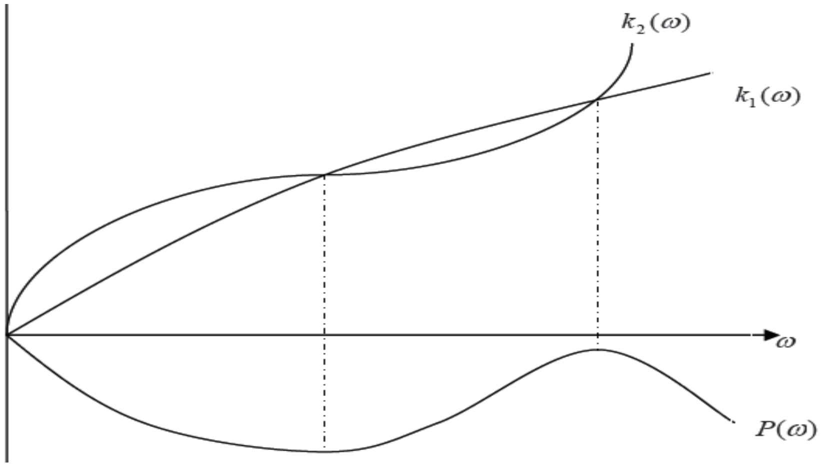
case $k_1(\omega) = k_2(\omega)$



Like 1-sector !

2-sector production:Harrod-Johnson diagram

Crossing factor intensities:



No factor-price equalization

2-sector production

- Factor price equalization holds if capital intensity schedules do not mot cross.
- When factor price equalization holds (assuming not specializaiton),

then free, costless trade in outputs makes tade in factors "redundant"