Lecture Notes #10

K. J. Arrow, "The Role of Securities in the Optimal Allocation of Risk-bearing", The Review of Economic Studies, Vol. 31, No. 2 (Apr., 1964), pp. 91-96

Uncertainty

Arrow-Debreu Contingent Claims

$s = \alpha, \beta \qquad h \in H, \qquad i = 1, 2, ..., l$

$Max\ \pi_h(\alpha)u_h(x_h(\alpha)) + \pi_h(\beta)u_h(x_h(\beta))$

$s.t.\ p(\alpha)x_h(\alpha) + p(\beta)x_h(\beta) = p(\alpha)\omega_h(\alpha) + p(\beta)\omega_h(\beta)$

$p = (p(\alpha), p(\beta))$ is CE if $\sum_H x_h(\alpha) = \sum_H \omega_h(\alpha)$

and $\sum_H x_h(\beta) = \sum_H \omega_h(\beta)$

Arrow Securities (Arrow Money)

$s = \alpha, \beta \qquad h \in H, \qquad i = 1, 2, ..., l$

Securities market meets before s is revealed.

Spot market for commodities meets after s is revealed

1 unit of $b_h(s)$ promises to pay h "one dollar" in state s.

$$Max\ \pi_h(\alpha)u_h(x_h(\alpha)) + \pi_h(\beta)u_h(x_h(\beta))$$

$$s.t. \sum_s p_b(s)b_h(s) = 0 \quad or \quad p_b(\alpha)b_h(\alpha) = -p_b(\beta)b_h(\beta)$$

$$p(s)x_h(s) = p(s)\omega_h(s) + b_h(s) \qquad s = \alpha, \beta$$

Equilibrium$\Rightarrow \sum_H b_h(s) = 0$, $\sum_H x_h(s) = \sum_H \omega_h(s)$ for $s = \alpha, \beta$
It follows that $p_b(\alpha) + p_b(\beta) = 1$

To show that a CE in AD is also a CE to AS (but not the reverse):

$$p(\alpha)x_h(\alpha) = p(\alpha)\omega_h(\alpha) + b_h(\alpha)$$

$$p(\beta)x_h(\beta) = p(\beta)\omega_h(\beta) - \frac{p_b(\alpha)b_h(\alpha)}{p_b(\beta)}$$

$$p(\alpha)[x_h(\alpha) - \omega_h(\alpha)] = b_h(\alpha)$$

$$-\frac{p(\beta)p_b(\beta)}{p_b(\alpha)}[x_h(\beta) - \omega_h(\beta)] = b_h(\alpha)$$

$b_h(\alpha)$ is slack variable, hence constraint can be written as:

$$p(\alpha)[x_h(\alpha) - \omega_h(\alpha)] + \frac{p(\beta)p_b(\beta)}{p_b(\alpha)}[x_h(\beta) - \omega_h(\beta)] = 0 \text{ or}$$

$$p_b(\alpha)p(\alpha)[x_h(\alpha) - \omega_h(\alpha)] + p(\beta)p_b(\beta)[x_h(\beta) - \omega_h(\beta)] = 0$$

Let $\widehat{p}(s) = p_b(s)p(s)$. Then

$$\widehat{p}(\alpha)[x_h(\alpha) - \omega_h(\alpha)] + \widehat{p}(\beta)[x_h(\beta) - \omega_h(\beta)] = 0$$

$\widehat{p}(s)$ is the $p(s)$ from AD. QED

But money can always be worthless in some or all states,

so there are many closed market equilibria in AS that are not

equilibrium in AD.

Irving Fisher

Replace $s = \alpha, \beta$ with $t = 1, 2$

Instead of Contingent Claims: perfect forward market

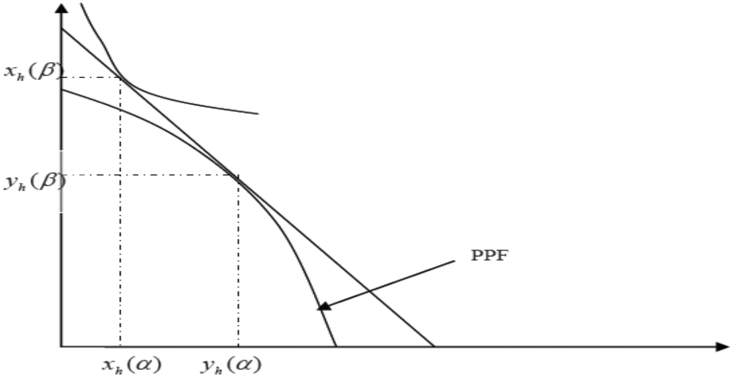
Instead of Arrow securities: perfect capital market

(i.e., perfect borrowing-and lending market)

Production

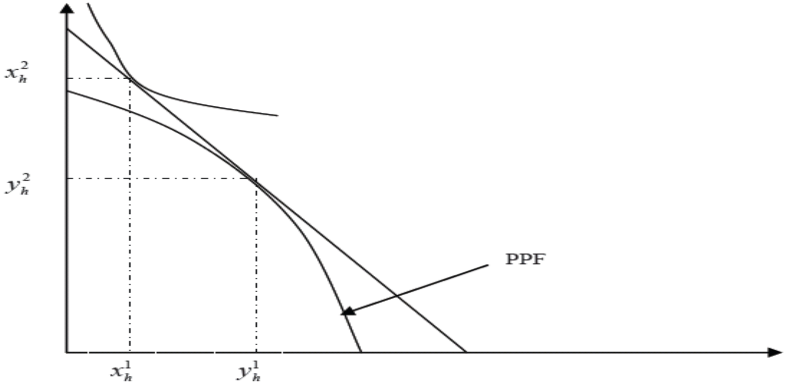
Profit max at contingent prices leads to $x_h(\beta) > y_h(\beta)$ and $y_h(\alpha) > x_h(\alpha)$

Not in general expected profit max



h is a lender. He produces $y_h^1 > x_h^1$ and consumes $x_h^2 > y_h^2$

$\frac{p^2}{p^1} = \frac{1}{1+r}$



Sunspots

Extrinsic Uncertainty

Preference:

$$v_h(x_h(\alpha), x_h(\beta); \pi_h(\alpha), \pi_h(\beta)) = v_h(x_h(\beta), x_h(\alpha); \pi_h(\beta), \pi_h(\alpha))$$

VNM satisfies this.

Endowments (symmetry): $\omega_h(\alpha) = \omega_h(\beta) \quad$ all h

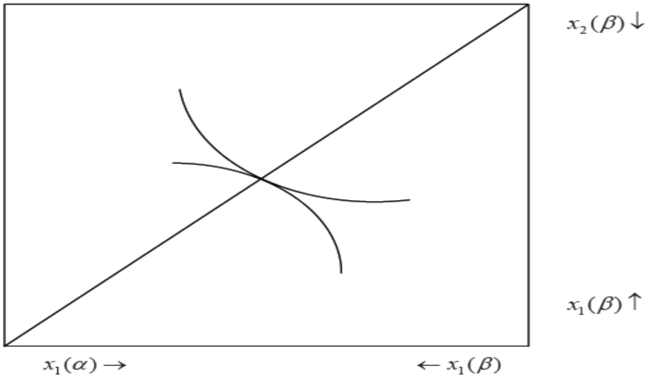
Production: $Y_f(\alpha) = Y_f(\beta) \quad$ all f

Edgeworth Box (square)

$$\pi_h(\alpha) = \pi(\alpha), \pi_h(\beta) = \pi(\beta)$$

$$\frac{\partial v_h/\partial x_h(\alpha)}{\partial v_h/\partial x_h(\beta)} = \frac{\pi(\alpha)\partial u/\partial x_h(\alpha)}{\pi(\beta)\partial u/\partial x_h(\beta)} \Rightarrow x_h(\alpha) = \partial x_h(\beta)$$

Sunspots do not matter.



Alternative Proof:

Suppose $x_h(\alpha) \neq \partial x_h(\beta)$ for some h

$v_h = \pi_h(\alpha)u(x_h(\alpha)) + \pi_h(\beta)u(x_h(\beta))$

Claim: $\overline{x}_h(s) = \pi_h(\alpha)x_h(\alpha) + \pi_h(\beta)x_h(\beta)$ is Pareto superior and feasible.

Hence SSE is not Pareto and, in this simple model, not a CE.

$$\begin{aligned}
(1)\ \sum_H \overline{x}_h(s) &= \sum_H [\pi_h(\alpha)x_h(\alpha) + \pi_h(\beta)x_h(\beta)] \\
&= \pi_h(\alpha)\sum_H x_h(\alpha) + \pi_h(\beta)\sum_H x_h(\beta) \\
&= \pi_h(\alpha)\sum_H \omega_h(\alpha) + \pi_h(\beta)\sum_H \omega_h(\beta) = \sum_H \omega_h(s)
\end{aligned}$$

$$\begin{aligned}
(2)\ v_h(\overline{x}_h(\alpha), \overline{x}_h(\beta)) &= \pi_h(\alpha)u(\overline{x}_h(\alpha)) + \pi_h(\beta)u(\overline{x}_h(\beta)) \\
&= u(\overline{x}_h(\alpha)) \\
&= u(\pi_h(\alpha)x_h(\alpha) + \pi_h(\beta)x_h(\beta)) \\
&> \pi_h(\alpha)u(x_h(\alpha)) + \pi_h(\beta)u_h(x_h(\beta))
\end{aligned}$$

Cass and K. Shell, "Do Sunspots Matter?" Journal of Political Economy, April 1983, 91(2): 193-227.

π_h differ. Sunspots always matter.