Lecture Notes #13
Bank Runs:


Bank Runs

0

learn type, withdrawal decision

left-over bank balance

deposit decision

1

second withdrawal decision

2
continuum of \textbf{ex-ante} identical consumers

\( y \) units of consumption

fraction \( \alpha \) are impatient

\( \alpha \) is uncertain, realization \( \alpha_1 \)

\( \bar{u} \) for best consumption opportunity

\( \beta \bar{u} \) for next-best \( \beta < 1 \)

utility of ”left-over” bank balance, \( u(\cdot) \)
\[ U_I(C^1_I, C^2_I) = \begin{cases} 
\bar{u} + u(C^1_I + C^2_I - 1) & \text{if } C^1_I \geq 1 \\
\beta \bar{u} + u(C^1_I + C^2_I - 1) & \text{if } C^1_I < 1 
\end{cases} \]  

(1)

and

\[ U_P(C^1_P, C^2_P) = \bar{u} + u(C^1_P + C^2_P - 1), \]

where \( C^t_I \) (resp., \( C^t_P \)) is the total withdrawal of an impatient (resp., patient) consumer from the bank in period \( t \). Specification (1) is based on the assumption that the consumers will always be able to afford their consumption opportunities in period 2, and that \( \bar{u} \) is high enough so that it is optimal to undertake available consumption opportunities. It is assumed that \( u \) is an increasing, smooth, and strictly concave function of terminal (or left-over) consumption, so we have \( u' > 0 \) and \( u'' < 0 \).
Let $f$ denote the probability density function for $\alpha$, the fraction of the consumers who become impatient, which is assumed to be continuous and have support $[0, \bar{\alpha}]$, where $\bar{\alpha} < 1$. In keeping with the assumption that consumers are ex ante identical, think about the following process: First, nature determines $\alpha$ according to $f$. Then, nature selects each particular consumer to be impatient with probability $\alpha$ and patient with probability $(1 - \alpha)$. A consumer’s type is her private information.
There are two constant-returns-to-scale technologies, an illiquid, higher-yield technology denoted as technology $i$ (or sometimes A), and a liquid, lower-yield technology denoted as technology $\ell$ (sometimes B). Investing 1 unit of period-0 consumption in technology $i$ yields $R_i$ units of consumption in period 2. Investing 1 unit of period-0 consumption in technology $\ell$ yields $R_\ell$ units of consumption if held until period 2, or 1 unit of consumption if harvested in period 1, $1 < R_\ell < R_i$. 
Consumers who choose period 1 are assumed to arrive in ”random order” (i. e. each position is ”equally probable”). Let $z_j$ denote the position of consumer $j$ in the queue. Because of the sequential service constraint, consumption must be allocated to consumers as they arrive to the head of the queue, as a function of the history of transactions up until that point. We further assume that consumer $j$’s withdrawal can only be a function of her position, $z_j$, and that she has an opportunity to refuse to withdraw and return without prejudice in period 2. The bank cannot keep track of how many consumers have refused. Let $\alpha_1$ denote the measure of consumers who have actually made a withdrawal in period 1. In period 2, the bank chooses how to divide its remaining resources between those who have withdrawn in period 1 and those who have not.
A contract specifies the fraction of a consumer’s endowment invested in technology \( \ell \), denoted by \( \gamma \); her withdrawal in period 1 as a function of her arrival position, denoted by \( c^1(z) \); and her withdrawal in period 2 from technology \( \ell \) investments as a function of \( \alpha_1 \) and whether the consumer made a withdrawal in period 1 or not, denoted respectively by \( c^2_I(\alpha_1) \) and \( c^2_P(\alpha_1) \). That is, a consumer who receives \( c^2_I(\alpha_1) \) from technology \( \ell \) investments receives a total withdrawal in period 2 of \( C^2_I(\alpha_1) = c^2_I(\alpha_1) + (1 - \gamma)R_iy \). Similarly, a consumer who receives \( c^2_P(\alpha_1) \) from technology \( \ell \) investments receives a total withdrawal in period 2 of \( C^2_P(\alpha_1) = c^2_P(\alpha_1) + (1 - \gamma)R_iy \). It is assumed that parameters are such that nonnegativity constraints \( C^2_I(\alpha_1) \geq 0 \) and \( C^2_P(\alpha_1) \geq 0 \) never bind.
For the mechanism to be feasible, all remaining resources must be distributed in period 2. The resource constraint is given by

$$\alpha_1 c_i^2(\alpha_1) + (1 - \alpha_1) c_P^2(\alpha_1) = [\gamma y - \int_0^{\alpha_1} c^1(z) \, dz] R_\ell. \quad \text{(RC)}$$

Thus, the space of deposit contracts or mechanisms $M$ is given by

$$M = \{ \gamma, c^1(z), c_i^2(\alpha_1), c_P^2(\alpha_1) \mid \text{Equation (RC) holds for all } \alpha_1 \}.$$
Two financial systems: (1) In the *separated financial system*, consumers place a fraction \((1 - \gamma)\) of their wealth in technology \(i\), whose return cannot be touched by the bank. In terms of resource constraint (RC), this is equivalent to imposing the additional constraints: \(c^2_P(\alpha_1) \geq 0\) and, more importantly, \(c^2_I(\alpha_1) \geq 0\). Combined with incentive compatibility, these additional constraints give rise to overinvestment in technology \(\ell\) and the possibility of bank runs. (2) In the *unified financial system*, the bank is able to invest in both technologies. This allows the bank more flexibility in smoothing consumption and preventing runs. For example, when \(\bar{\alpha}\) consumers arrive in period 1 (the worst case scenario), the bank can liquidate all of its technology \(\ell\) holdings, but differentially reward consumers from technology \(i\) in period 2. Consumers who arrive in period 1 might receive less than \((1 - \gamma)R_iy\) in period 2, while consumers who wait might receive more than \((1 - \gamma)R_iy\). In terms of resource constraint (RC) this is equivalent to allowing \(c^2_P(\alpha_1)\), or, more importantly, \(c^2_I(\alpha_1)\), to be negative.
**Definition:** Consider either a unified financial system or a separated financial system, and a contract $m \in M$. Then the post-deposit game is said to have a *run equilibrium* if there is a Bayes-Nash equilibrium in which all consumers arrive in period 1 and a positive measure of patient consumers withdraw in period 1.
The Unified Bank System (the Unrestricted Bank)

- Step 1: Analyze the Post-Deposit Game (as in DD).
- Step 2: Solve for optimal contract assuming: If it is not to the advantage of a patient depositor to run when others do not run, then there will not be a run.
We restrict attention to environments in which it is beneficial to provide for urgent consumption opportunities whenever the resources are available. It is then desirable that the impatient consumers choose period 1 and the patient consumers choose period 2 for making their urgent withdrawals. Also, impatient consumers unable to withdraw in period 1 and patient consumers should take advantage of their consumption opportunities in period 2. Since the amount of a withdrawal in period 1 greater than one unit would be stored for period 2, there is no reason for the bank to provide more than one unit in period 1; hence we have $y \gamma \leq \bar{a}$, and we can restrict our search to contracts in which we have

$$c^1(z) = 1 \quad \text{for } z \leq \gamma y.$$  \hspace{1cm} (3)
Given (3) and the fact that patient consumers wait until period 2, the ex ante welfare $W$ is given by

$$W = \int_0^{\gamma y} [\bar{u} + (1 - \alpha)u((1 - \gamma)yR_A + c_P^2(\alpha) - 1)$$

$$+\alpha u((1 - \gamma)yR_A + c_i^2(\alpha))] f(\alpha) d\alpha$$

$$+ \int_{\gamma y}^{\bar{x}} [(1 - \alpha + \gamma y)\bar{u} + (\alpha - \gamma y)\beta \bar{u}$$

$$+(1 - \alpha)u((1 - \gamma)yR_A + c_P^2(\alpha) - 1)$$

$$+(\alpha - \gamma y)u((1 - \gamma)yR_A + c_P^2(\alpha) - 1)$$

$$+\gamma yu((1 - \gamma)yR_A + c_i^2(\alpha)) f(\alpha) d\alpha$$

(4)
where \( f_P(\alpha) = \frac{(1 - \alpha)f(\alpha)}{\int_0^{\alpha} (1 - a)f(a)da} \).
Thus, we have

\[
\int_0^{\bar{\alpha}} u(c_P^2(\alpha) + (1 - \gamma)yR_A - 1)f_P(\alpha)\,d\alpha \\
\geq \int_0^{\gamma y} u(c_i^2(\alpha) + (1 - \gamma)yR_A)f_P(\alpha)\,d\alpha \\
+ \int_{\gamma y}^{\bar{\alpha}} (\gamma y/\alpha)u(c_i^2(\alpha) + (1 - \gamma)yR_A) \\
+ (1 - \gamma y/\alpha)u(c_P^2(\alpha) + (1 - \gamma)yR_A - 1)f_P(\alpha)\,d\alpha.
\] (5)

Resource constraint (2) can be simplified to yield

\[
\alpha_1 c_i^2(\alpha_1) + (1 - \alpha_1)c_P^2(\alpha_1) = (\gamma y - \alpha_1)R_B \quad \text{if } \alpha_1 \leq \gamma y
\] (6)

\[
\gamma yc_i^2(\alpha_1) + (1 - \gamma y)c_P^2(\alpha_1) = 0 \quad \text{if } \alpha_1 > \gamma y.
\]
Profit-maximizing perfectly-competitive bank chooses the contract so as to:

\[
\max_{\gamma, c_t^2(\alpha_1), c_p^2(\alpha_1)} W
\]

subject to (5) and (6).  \hfill (7)
**Theorem:** The so-called “optimal contract” for the unified system satisfies $\gamma y < \bar{\alpha}$. The “first” $\gamma y$ impatient consumers to arrive are fully served by the bank in period 1. There is a positive probability that $\alpha > \gamma y$ holds, in which case $(\alpha - \gamma y)$ impatient consumers are rationed. Patient consumers do not withdraw in period 1, and we have full consumption smoothing, i.e.,

$$c_i^2(\alpha_1) = c_P^2(\alpha_1) - 1$$

for all $\alpha_1 \leq \gamma y$. (8)
Proof: Given \( \gamma \), the functions \( \{ c_i^2(\alpha_1), c_P^2(\alpha_1) \} \) that maximize \( W \) subject only to resource constraint (6) entail full consumption smoothing (8). However, the allocation defined by (6) and (8) also satisfies the incentive compatibility constraint (5), and therefore solves the more tightly constrained problem (7). Plugging (6) and (8) into the expression for \( W \), and differentiating with respect to \( \gamma \), we have

\[
\left( \frac{\partial W}{\partial \gamma} \right)_{\gamma=\bar{\alpha}/y} = \int_0^{\bar{\alpha}} y(R_B - R_A)u'[c_P^2(\alpha) - 1 + (y - \bar{\alpha})R_A]f(\alpha)d\alpha < 0.
\]

Clearly, any contract for which we have \( \gamma y > \bar{\alpha} \) is inferior to the one characterized by (6) and (8), since the former provides fewer resources available in period 2, with no compensating advantage in terms of consumption smoothing in period 2 or provision of consumption in period 1.
The proof shows that the optimal contract offers complete consumption smoothing, according to (6) and (8). Since the patient consumers receive the same consumption, whether they arrive in period 1 or period 2, it is obviously incentive compatible. The simple form of the consumption opportunities ensures that the optimal contract has a simple, realistic solution in which there is full, but not partial, suspension of convertibility.
**Theorem:** There is an "optimal contract" for the unified system. For any "optimal contract", the corresponding allocation is socially optimal, maximizing $W$ subject only to the resource constraint, (6). Assuming that a patient depositor will choose not to run when indifferent between running and not running, there is an optimal contract that does not have a run equilibrium.
Proof: First, note that a solution to (7) must exist. Given \( \gamma \), a sufficient condition for \( \{ c^2_I(\alpha_1), c^2_P(\alpha_1) \} \) to solve problem (7) is for (6) and (8) to hold. The specification of consumption for \( \alpha_1 > \gamma y \) does not affect the objective or the incentive compatibility constraint. Plugging (6) and (8) into the expression for \( W \), problem (7) can be transformed into an equivalent unconstrained problem of choosing \( \gamma \) to maximize \( W \), which must have a solution satisfying \( \gamma y < \bar{\alpha} \). Because (6) and (8) imply (5), it follows that the optimal contract is socially optimal.

Construct the contract, \((\gamma, c^2_I(\alpha_1), c^2_P(\alpha_1))\), as follows. Let \( \gamma \) be as in the solution to (7). Consumptions are determined by the resource equation, (6), and

\[
c^2_I(\alpha_1) = c^2_P(\alpha_1) - 1
\]

for all \( \alpha_1 \).

It follows that patient consumers are indifferent between withdrawing in period 1 and waiting. Under the assumption that a patient consumer will choose not to run when indifferent between running and not running, the "optimal contract" does not have a run equilibrium.
The Separated System

In the separated system, ex-ante welfare, the incentive compatibility constraint, and the resource constraint are as given in expressions (4), (5), and (6). The restriction that the bank cannot gain access to investments of technology $A$ is expressed simply as follows:

$$c^2_I(\alpha_1) \geq 0 \text{ and } c^2_P(\alpha_1) \geq 0 \text{ for all } \alpha_1.$$  (10)

Notice that constraints (6) and (10) imply $c^2_I(\alpha_1) = c^2_P(\alpha_1) = 0$ for $\alpha_1 > \gamma y$. If all of the technology $B$ investments are liquidated in period 1, then withdrawals from the bank must be zero in period 2. An optimal contract under the separated system is a solution to the following planner’s problem:

$$\max_{\gamma, c^2_I(\alpha_1), c^2_P(\alpha_1)} W$$
subject to (5), (6), and (10).  (11)
Lemma: Any "optimal contract" in the separated system, which solves problem (11), satisfies $(c^2_P(\bar{\alpha}))^* < 1$.

Hence there is always a run equilibrium, since $1 > \bar{\alpha}$. If $\gamma y > \bar{\alpha}$, it would be a over-investment in liquidity. If $\gamma y = \bar{\alpha}$, it would also be overinvestment against low probability needs.
Proof: Suppose instead that \((c_\bar{P}^2(\bar{\alpha}))^* > 1\) holds. Since resources remain in period 2, it follows that \(\bar{\alpha} \leq \gamma^* y\) holds. Therefore, it is possible to increase welfare by reducing \(\gamma\) and achieving full consumption-smoothing, 
\((c_\bar{P}^2(\alpha_1))^* = 1 + (c_i^2(\alpha_1))^*\) for all \(\alpha_1\). It is easy to see that incentive compatibility and nonnegativity are satisfied, contradicting the fact that \(m^*\) solves (11).

Now suppose that \((c_\bar{P}^2(\bar{\alpha}))^* = 1\) holds. We must have \((c_i^2(\bar{\alpha}))^* = 0\), or else (just as in the previous case) we can increase welfare by reducing \(\gamma\), while maintaining full consumption-smoothing. It follows that we must have 

\[
(c_\bar{P}^2(\alpha_1))^* - (c_i^2(\alpha_1))^* \geq 1
\]

(12) for almost all \(\alpha_1\). Otherwise, for a positive-measure set of realizations of \(\alpha_1\), \(c_\bar{P}^2(\alpha_1)\) can be increased and \(c_i^2(\alpha_1)\) can be reduced to satisfy the resource and nonnegativity constraints, which increases welfare and relaxes the incentive compatibility constraint.
If inequality (12) is strict for a positive-measure set of realizations of $\alpha_1$, then welfare can be feasibly increased by choosing $(c_P^2(\alpha_1))^*$ and $(c_I^2(\alpha_1))^*$ to satisfy full consumption-smoothing (where (12) holds as an equality) and the resource constraint, (6). Therefore, we have for all $\alpha_1$,

$$(c_P^2(\alpha_1))^* - (c_I^2(\alpha_1))^* = 1. \tag{13}$$

Treating $\gamma$ as a parameter, and solving the equations (6) and (13) for consumptions, we can define welfare as a function of $\gamma$, $W(\gamma)$. Because $m^*$ solves (11), $W(\gamma)$ must be maximized at $\gamma = \gamma^*$. Applying the envelope theorem, one can show:

$$W'(\gamma^*) = y(R_B - R_A) \int_0^{\bar{\alpha}} u'[\gamma R_A + (\gamma y - \alpha) R_B + \alpha - 1] f(\alpha) d\alpha <$$

It follows that reducing $\gamma$ improves welfare, contradicting the fact that $\gamma^*$ is part of the optimal contract, $m^*$. 
**Theorem:** Any "optimal contract" in the separated financial system has a run equilibrium.
**Theorem (Overinvestment in the Liquid Asset):** If $\bar{\alpha} < 1/R_B$ holds, then any optimal contract for the separated financial system does not ration consumers in period 1 in the no-run equilibrium, and invests more in technology $B$ than any optimal contract for the unified financial system.

This is a weak sufficient condition for over-investment in liquidity. (The more liquidity, the less economic growth.)
Numerical Example: The unified system

Example 3.3:

\[ y = 10, \ u(c) = 100 \log(c) - 249, \ \bar{u} = 20, \ RA = 1.1, \ \beta = 0.7, \]

uniform distribution with \( \bar{\alpha} = 0.5 \):

\[ f(\alpha) = \begin{cases} 
2 & \text{for } \alpha \in [0, 0.5] \\
0 & \text{otherwise.} 
\end{cases} \quad (9) \]

For the unified financial system, we compute \( \gamma \), the proportion of wealth invested in technology \( B \) and the ex-ante welfare of consumers \( W \) for different values of the interest factor \( R_B \) on the liquid technology. For \( R_B = 1.05 \), we have \( \gamma = 0.04544 \) and \( W = 0.8942 \); for \( R_B = 1.08 \), we have \( \gamma = 0.04807 \) and \( W = 0.9599 \). An increase in \( R_B \) increases welfare, due to the higher yield on technology \( B \) investments, and increases \( \gamma \), because reducing the probability of rationing consumers in period 1 is now less costly, because the gap between the yields in technologies \( A \) and \( B \) has been reduced.
Numerical Example: The separated system (The Glass-Steagall Bank)

Example 4.4: Consider the parameter values specified in Example (3.3). We also specify $R_B = 1.08$.

It can be shown that the "optimal contract" satisfies $(c^2_f(\alpha_1))^* = 0$ for all $\alpha_1$. Since Theorem (4.3) applies, it follows from (6) that we have

$$
(c^2_P(\alpha_1))^* = \frac{(10\gamma - \alpha_1)1.08}{1 - \alpha_1}.
$$

(16)

Finding the "optimal contract" now reduces to finding the value of $\gamma$ that maximizes welfare subject to the incentive compatibility constraint. The optimal $\gamma$ will cause the incentive compatibility constraint to hold with equality, yielding: $\gamma^* = 0.09445$ and $W^* = 0.8688$. Comparing the "optimal contract" in the unified system to the "optimal contract" in the separated system, we see that technology $B$ investment in the separated system is nearly double that in the unified system.
Problem Set #12

Fix $R = 1.08$. Calculate the loss (in terms of the endowment $y$) in moving from unrestricted banking to restricted banking. In other words, calculating the value $y$ in the unrestricted system that yields the same welfare as $y = 10$ in the restricted system.
The pre-deposit game, sunspots, and an example

The run equilibrium to our ”optimal contract”–like the run equilibrium in Diamond and Dybvig (1983)–is not really an equilibrium to the pre-deposit game, because consumers would not deposit their funds if they knew that a run would take place. See Postlewaite and Vives (1987). Diamond and Dybvig suggest that a run could take place in equilibrium with positive probability, triggered by “sunspots,” as long as the probability of the run is sufficiently small. Cooper and Ross (1998) and Peck and Shell (2003) formalize this notion. Consider our example, defined by the parameters in (9), and $R_B = 1.08$. From (16), we can calculate the optimal $\gamma$ to avoid runs, $\gamma^{**} = 0.09630$. By comparing welfare in the optimal contract that avoids runs and welfare in the optimal contract that tolerates runs, one can show that the cutoff propensity to run is 0.005521. If the sunspot-driven propensity to run is less than 0.5521%, then it is better to tolerate the unlikely event of a run than to increase technology $B$ investment to prevent runs. Peck and Shell (2003) put ”runs” back into the bank-runs literature!