Lecture Notes #3: Economic Growth

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Economic Growth: One-Sector Technology

- $Y = C + Z = F(K, L)$
- $C \geq 0,\ Z \geq 0$
- $\dot{K} = F(K, L) - C - \mu K,\ \mu > 0$
- $\dot{L} = nL,\ n > 0$
- $\lambda = \mu + n > 0$
- $y = \frac{Y}{L},\ c = \frac{C}{L},\ z = \frac{Z}{L},\ k = \frac{K}{L}$

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Log differentiatng $k = \frac{K}{L}$:

$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$

% change in $k =$%change in K- % change in L

$\frac{\dot{k}}{k} = \frac{F(K,L) - C - \mu K}{K} - n$

$\dot{k} = \frac{(K/L)(F(K,L) - C - \mu K)}{K} - nk$

$\dot{k} = f(k) - c - \lambda k$

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- $\dot{k} = 0$ defines the steady state (in which K and L are growing at the same rate).
  $\dot{k} = 0$ is also the stationary solution to the DE $\dot{k} = 0$
- In the steady state we have:
  $c = f(k) - \lambda k$
  Hence for steady-state production c is a function of k :
  $c(k) = f(k) - \lambda k$
- We next study by graphing and calculus $c(k)$
  We will find the golden rule k discovered by Edmund S. (Ned) Phelps

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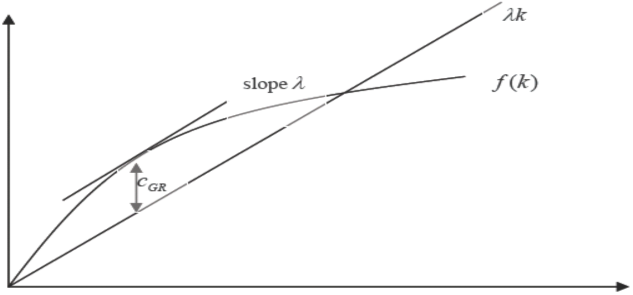

- Golden Rule Capital Labor Ratio, k_{GR} , maximizes consumption across steady states.
- $f'(k_{GR}) = \lambda$
- $r = f'(k) - \mu$

$r_{GR} = n$ rate of interest=growth of labor

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- $c(k) = f(k) - \lambda k$
  $\frac{dc}{dk} = f'(k) - \lambda$
  $f'(k_{GR}) - \mu = n$
  $r_{GN} = N$
- $\frac{d^2c}{dk^2} = f'' < 0$ 2nd order condition
- Modified Golden Rule:
  $f'(k_{MGR}) = \delta + \lambda$
  David Cass

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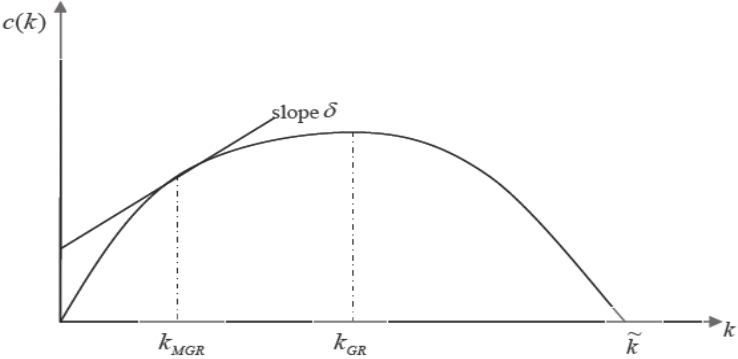
another graph



- δ positive in this graph
- $\tilde{k}$ is the maximal sustainable capital-labor ration: $c(\tilde{k}) = 0$ over-saving !

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Meaning of GR

- Not necessarily "optimal"
  - Initial k is not given
  - What use is it to advise country with very low k to choose k_{GR} ?
  - Why should one want to go to a steady state even if feasible ?
  - optimal growth later !
- Efficiency:
  While $\{k(t)\}$ always small can be efficient, $\{k(t)\}$ "too large" will be inefficient. Oversaving ! Because of infinite horizon

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- Phelps-Koopmans Inefficiency Theorem
  If $\{k(t)\}$ is forever bounded above k_{GR}, then the program $\{k(t), c(t)\}$ is inefficient in the sense that $c(t)$ can be increased in some intervals without decreasing $c(t)$ elsewhere
- Definitions:
  Consider the feasible program:

  $$\{c(t) \mid c(t) \leq f(k(t)) - \lambda k(t) - \dot{k}(t),\ k(0) \leq k_0,\ t \in [0, \infty]\}$$

  The program is said to be inefficient if there exist $c'(t)$ and $k'(t)$ with the properties:

  $$\{c'(t) \mid c'(t) \leq f(k'(t)) - \lambda k'(t) - \dot{k}'(t),\ k'(0) \leq k_0,\ t \in [0, \infty]\}$$

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and

$c'(t) \geq c(t)$ for $t \in [0, \infty]$

while

$c'(t) > c(t)$ for some interval $[t_1, t_2] \subset [0, \infty]$, $t_1 < t_2$

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Proof of the KP Theorem:
assume that after some date $t^* \geq 0$, $k(t) > k_{GR} + \varepsilon$ for some (small) $\varepsilon > 0$. Then we can construct $k'(t)$ from $k(t)$
$k'(t) = k(t)$ for $t \in [0, t^*]$
$k'(t) = k(t) - \varepsilon$ for $t \geqslant t^*$
$c'(t) = f(k'(t)) - \lambda k'(t) = f(k(t) - \varepsilon) - \lambda(k(t) - \varepsilon) > c(t)$

Descriptive Growth (Solow)

Reference:

- R. M. Solow, "A Contribution to the Theory of Economic Growth", Quarterly Journal of Economics, February 1956

compositive of output (C, Z) determined be exogenously
given saving functions, $s > 0$
$Z = sY$
$C = (1 - s)Y$
$\dot{k} = f(k) - \lambda k$
$k(0) = k_0 > 0$

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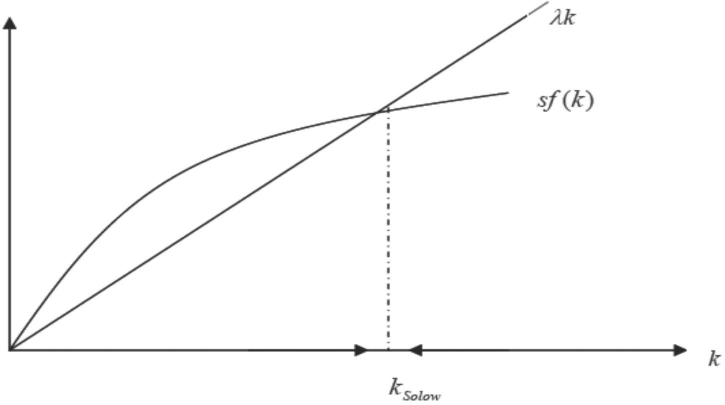
Phase Line



More generally, Phase Diagrams for all dynamical systems including differential equations and difference equations

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- Solow model is locally stable and globally stable is we assume $k_0 > 0$.
  0 is unstable
- History independence. Neither a good thing nor a bad thing.
  QWERTY
  Massachusetts vs New York
  Railroad crossing

Optimal Growth

Reference:

- 1. D. Cass, "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies, July 1965.
- 2. Shell's Varenna Lectures (1969)

Problem will be to choose $\{0 \leq s(t) \leq 1\}$ to maximize
$\int_0^T U(c(t))e^{-\delta t}dt$
where $c(t) = (1 - s(t))f(k(t))$
$\dot{k} = s(t)f(k(t)) - \lambda k(t)$
where $k(0) = k_0$ and $k(T) = k_T$,
First also assume (for the moment) that $T < \infty$ is finite

Optimal Growth

Calculus of Variations and the Euler Equation

- Problem is to find a best function, not merely a best point.
- Consider $x(t)$ defined on $[t_0, t_1]$ and $x(t_0) = x_0$, $x(t_1) = x_1$ and to extremize the integrand $\int_{t_0}^{t_1} \Phi(x(t), \dot{x}(t), t))dt$
- Euler's necessary condition is that $\Phi_x = \frac{d}{dt}\Phi_{\dot{x}}$ be satisfied.

Optimal Growth

Derivation:

Consider $y(t,\varepsilon) = x(t) + \varepsilon u(t)$ to be a variation of the function $x(t)$.

$$V(\varepsilon) = \int_{t_0}^{t_1} \Phi(x(t) + \varepsilon u(t), \dot{x}(t) + \varepsilon \dot{u}(t), t))dt$$

$$V'(\varepsilon) = \int_{t_0}^{t_1} (u(t)\Phi_y + \dot{u}(t)\Phi_{\dot{y}})dt = 0$$

Integrate by parts:

$$\int_{t_0}^{t_1} \dot{u}(t)\Phi_{\dot{y}}\, dt = u\Phi_{\dot{y}} \,\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} u(\tfrac{d}{dt}\Phi_{\dot{y}})dt$$

$$= 0 - 0 - \int_{t_0}^{t_1} u(\tfrac{d}{dt}\Phi_{\dot{y}})dt = -\int_{t_0}^{t_1} u(\tfrac{d}{dt}\Phi_{\dot{y}})dt$$

Optimal Growth

So $\int_{t_0}^{t_1} u(\Phi_y - (\frac{d}{dt}\Phi_{\dot{y}}))dt = 0$.
This must hold for every function $u(t)$.
Hence we have $\Phi_y = (\frac{d}{dt}\Phi_{\dot{y}})$ or $\Phi_x = \frac{d}{dt}\Phi_{\dot{x}}$

Optimal Growth

Apply to one-sector RCK problem assuming, one can eat capital and further no corner conditions are involved.

$$\int_0^\infty U(f(k) - \lambda k - \dot{k})e^{-\delta t}dt$$

$$\Phi_k = U' \cdot (f'(k) - \lambda)e^{-\delta t}$$

$$\Phi_{\dot{k}} = -U'e^{-\delta t}$$

Optimal Growth

Let $q = U'$ be the current demand price k and $qe^{-\delta t}$ the present demand price of k . Then we have:

$q \cdot (f'(k) - \lambda)e^{-\delta t} = \frac{d}{dt}(-qe^{-\delta t}) = -\dot{q}e^{-\delta t} + \delta q e^{-\delta t}$

$\dot{q} = q(f'(k) - \lambda - \delta)$

$\dot{k} = f(k) - \lambda k - c(q)$

where $U'(c) = q$

Hence $\frac{dq}{dc} = U''(c) < 0$ and $\frac{dc}{dq} = \frac{1}{U''(c)} < 0$

Optimal Growth

Stationeries:
$\dot{q} = 0$ yields $f'(k) = \delta + \lambda$ call root k^* MGR k
$\dot{k} = 0$ yields
$c(q) = f(k) - \lambda k$
$c'(q)\frac{dq}{dk} = f'(k) - \lambda$
$\frac{dq}{dk} = (f'(k) - \lambda)U''$ achieves minimum at k_{GR}
Phase diagram next:

Optimal Growth

RCK phase diagram