
Diamond Model

2-period lives:

work and consume in the first period

retire and consume in the second period
\( L_t : \) workers born in \( t \)

\[
L_t = L_0(1 + n)^t \quad Y_t = F(K_t, L_t)
\]

\[
C^1_t = c^1_t L_t \quad \quad C^2_t = c^2_t L_{t-1}
\]

\[
C_t = C^1_t + C^2_t \quad \quad Y_t + K_t = K_{t+1} + C_t
\]
Steady State

\[ K_{t+1} = (1 + n) K_t \]

\[ y_t - nk_t = C_t / L_t = c^1_t + c^2_t / (1 + n) \]

Planner Problem

Max \( u(c^1, c^2) \)

subject to \[ c^1 + c^2 / (1 + n) = y - nk \]

Step 1: \[ \frac{\partial y}{\partial k} = n \] produce at golden-rule level

Step 2: Planner: \[ \frac{\partial u}{\partial c^1} = (1 + n) \frac{\partial u}{\partial c^2} \]
Market Economy:

\[ S_t = s_t L_t = L_t s(w_t, r_{t+1}) \]

\[ r_{t+1} = f'(k_{t+1}) \quad k_{t+1} = \frac{K_{t+1}}{L_{t+1}} \]

\[ r_{t+1} = f'(S_t/L_{t+1}) = f'(s(w_t, r_{t+1})/(1+n)) \]
Example:

\[ u(c^1, c^2) = \beta \log c^1 + (1 - \beta) \log c^2 \]

\[ s = (1 - \beta)w \]

\[ r_{t+1} = f'\left((1 - \beta)w_t/(1+n)\right) \quad y = Ak^\alpha \]

\[ r_{t+1} = \alpha A[(1 - \beta)w_t/(1 + n)]^{\alpha-1} \]

\[ w_t = (1 - \alpha)\alpha^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} r_t^{\frac{\alpha}{1-\alpha}} \]

\[ r_{t+1} = \left(\frac{\alpha(1+n)}{(1-\beta)(1-\alpha)}\right)^{1-\alpha} r_t^\alpha \]

\[ \lim_{{t \to \infty}} r_t = r^* = \frac{\alpha(1+n)}{(1-\beta)(1-\alpha)} \]

Unless \( n = \frac{\alpha}{(1-\beta)(1-\alpha)} - \alpha \), \( r^* \) is not at GR level.
Internal Debt:

\[
\delta = \Delta / L \quad \quad \delta_t = \delta_{t+1} = \text{cons} = \delta
\]

\[
\Delta_{t+1} = (1 + n) \Delta_t
\]

Savings:

\[
s(w_t - (r_t - n)\delta, r_{t+1}) \quad S_t = K_{t+1} + \Delta_{t+1}
\]

\[
\frac{s_t}{1+n} = k_{t+1} + \delta
\]

\[
r_{t+1} = f'(s(w_t - (r_t - n)\delta, r_{t+1}) / (1 + n) - \delta)
\]