

Extrinsic Uncertainty Revisited*

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This paper is devoted to extending the concept of extrinsic uncertainty introduced by Cass and Shell for von Neumann-Morgenstern utility functions to a class of ordinal utility functions. It is shown that the important result that extrinsic uncertainty does not matter in a pure exchange economy extends to the more general framework. *Journal of Economic Literature* Classification Numbers: 021, 022, 026.

1. INTRODUCTION

The relationship between uncertainty and equilibria as a function of market participation has been investigated by Cass and Shell [3]. One of their results is that, in the case of common beliefs on the states of nature, "extrinsic" uncertainty does not matter when there is full market participation. Their definition of extrinsic uncertainty amounts to taking expected values over states of nature of utility functions defined for certainty. Their approach fits, therefore, into the framework of Chapter 7 of Debreu's *Theory of Value* [4], but only as a special case, the one that corresponds to von Neumann-Morgenstern utility functions.

The aim of this paper is to show that the idea of extrinsic uncertainty can be formulated with the level of generality of Chapter 7, the results of Cass and Shell remaining true with the more general definition. Furthermore, the approach followed in this paper stresses the role played by ideas of symmetry and of invariance when one deals with extrinsic uncertainty. The proofs given here may seem somewhat intricate. This is only a superficial impression resulting from the careful treatment of most details, the basic lines of the proofs being on the contrary extremely simple. This type of argument may appear unusual; it is in fact widely used in several fields of mathematics closely related to group theory.

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2. THE CONCEPT OF EXTRINSIC UNCERTAINTY

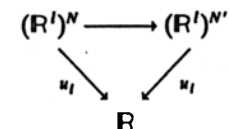
We denote by l the number of physical commodities, by m the number of consumers. Each consumer is characterized by his initial endowments and by his preferences; we assume that preferences are represented by utility functions.

Let N be any set of states of nature. Then our approach is to assume that preferences are defined on the contingent commodity space $(\mathbb{R}^l)^N$ for any set N . This implies that we have different preferences, defined on different contingent commodity spaces for different sets N . A crucial feature of the idea of extrinsic uncertainty is provided by the axiomatic description of the relationships existing between the preferences on these various contingent commodity spaces depending on the relationships relating the sets of states of nature. Among the important examples of varying sets of nature, we find the construction of events through the decomposition of given events into more elementary events with arbitrarily small objective probability; in other words, we are always able to define, given a set of states N , a new set N' of states of nature such that N consists of the elements of a partition of N' , the probability of every elementary event in N' being arbitrarily small (see the comments preceding Axiom 3 for such constructions). In fact, we are considering a nested family of sets representing the states of nature.

Let N be a given set of states of nature; let $v \in N$. We denote by $x(v)$ the commodity bundle $x \in \mathbb{R}^l$, the availability of which is contingent to the realization of the state $v \in N$. If n denotes the number of elements of N , the contingent commodity space \mathbb{R}^l consists of the n -tuples $(x(v_1), x(v_2), \dots, x(v_n))$. We find it convenient to identify a contingent bundle, i.e., an n -tuple, to a mapping $x: N \rightarrow \mathbb{R}^l$, as the contingent commodity space becoming the space of mappings from N into \mathbb{R}^l , denoted $(\mathbb{R}^l)^N$.

Let N and N' be two sets of states of nature, N' being a refinement of N ; in other words N consists of subsets of N' defining a partition. We can associate with the partition N of N' an equivalence relation \sim on N' , the equivalence classes of which are the elements of N . We assume that consumer i ($i = 1, 2, \dots, m$) has utility functions, both denoted u_i , defined respectively on the commodity spaces $(\mathbb{R}^l)^N$ and $(\mathbb{R}^l)^{N'}$. Note that $(\mathbb{R}^l)^N$ is naturally embedded in $(\mathbb{R}^l)^{N'}$ in the following way: The mapping $x: N \rightarrow \mathbb{R}^l$ defines a mapping $N' \rightarrow \mathbb{R}^l$ by associating with $v \in N'$ the value $x(\bar{v})$, where \bar{v} denotes the equivalence class of v for \sim . The axiom of compound events relate the values of u_i defined on $(\mathbb{R}^l)^N$ and $(\mathbb{R}^l)^{N'}$:

AXIOM 1 (Compound events). *The diagram is commutative for the natural embedding of $(\mathbb{R}^l)^N$ into $(\mathbb{R}^l)^{N'}$.*

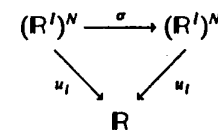


Our formulation shows how natural a requirement is the axiom of compound events. It is very similar to the assumption on the reduction of compound lotteries of Luce and Raiffa [6, assumption 2, p. 26].

We are now ready to formulate the idea of extrinsic uncertainty and to derive its main consequences. Assume now that N is a set of equiprobable states of nature. Let σ be any permutation (i.e., bijection) of N . Then σ operates on the set of mappings $(\mathbb{R}^l)^N$ by associating $x_0 \sigma: N \rightarrow \mathbb{R}^l$ with $x: N \rightarrow \mathbb{R}^l$. We denote by x^σ the image of x by σ . We then have:

AXIOM 2 (Symmetry of equiprobable events). *If N is a set of equiprobable states of nature, we say that uncertainty is extrinsic with respect to the utility function $u_i: (\mathbb{R}^l)^N \rightarrow \mathbb{R}$ if we have $u_i(x) = u_i(x^\sigma)$ for any $x \in (\mathbb{R}^l)^N$ and any permutation σ of N .*

In fact, σ defines by the formula $x \mapsto u_i(x^\sigma) = u_i^\sigma(x)$ a mapping $u_i \mapsto u_i^\sigma$ from $(\mathbb{R}^l)^N$ into itself. Axiom 2 can be reformulated as saying that the diagram



is commutative.

For example, in the case $N = \{1, 2\}$, Axiom 2 implies $u_i(x(1), x(2)) = u_i(x(2), x(1))$ for any $x(1) \in \mathbb{R}^l$ and $x(2) \in \mathbb{R}^l$. Once events are equiprobable, consumer i does not really mind how contingent bundles depend on the labelling of the equiprobable states of nature.

Consider now an arbitrary set of states of nature N and assume that the objective probability π_i of event i is a rational number for every $v_i \in N$; π_i can be given the form $\pi_i = \alpha_i / \beta$, where α_i and β are integers. One can decompose the event v_i into α_i equiprobable (with probability $1/\beta$) events by considering the event consisting simultaneously of v_i and of the independent outcome of a lottery taking integer values between 1 and α_i . This defines a set N' of equiprobable states of nature. This construction is possible only if the objective probabilities $\pi = (\pi_i)$ are all rational numbers. To be able to consider cases where the probabilities $\pi = (\pi_i)$ may be irrational numbers, we must introduce a continuity assumption with respect to probabilities. Let P

be the positive simplex of \mathbb{R}^n ; then $\pi = (\pi_i)$ belongs to P since $\sum \pi_i = 1$. From now on, we consider only smooth utility functions (i.e., they are differentiable up to any order). The set of smooth mappings $(\mathbb{R}^l)^N \rightarrow \mathbb{R}$ is endowed with the Whitney topology [5], the most natural topology in that space. We then have:

AXIOM 3 (Continuity with respect to objective probabilities). *Let $(N, \pi) \in N \times P$ be given; then there exists a neighborhood U of π in P such that the utility function u_i is defined on $(\mathbb{R}^l)^{N \times U}$ and depends continuously on $\pi \in U$ for $i = 1, 2, \dots, m$.*

Since the set of rational numbers is dense in the set of real numbers, Axiom 3 will enable us, when $\pi = (\pi_i)$ contains irrational numbers, to consider some π'_i rational numbers.

Remark. It is clear that any von Neumann–Morgenstern utility function satisfies Axioms 1–3. Nevertheless, linearity with respect to the probability of the states of nature is an important feature of von Neumann–Morgenstern utilities that is not necessary in the framework considered here. For example, any monotone transformation of a von Neumann–Morgenstern utility function defines preferences satisfying Axioms 1–3 and does not belong to the class of von Neumann–Morgenstern utilities.

3. FINITE PURE-EXCHANGE ECONOMIES AND EXTRINSIC UNCERTAINTY

We consider a finite pure-exchange economy (recall there are m consumers, l physical commodities, and a set of states of nature N). We assume that Axioms 1, 2, and 3 are satisfied for any element belonging to any nested family of states of nature containing N , a situation we summarize by saying that there is extrinsic uncertainty with respect to every consumer (of course, Axiom 2 is valid only for sets N' consisting of equiprobable events). In fact, the key to the analysis of pure-exchange economies with extrinsic uncertainties is the remarkably simple structure of the Pareto efficient allocations in these economies. We recall that an allocation $x = (x_1, x_2, \dots, x_m)$ consists of the contingent bundles $x_i: N \rightarrow \mathbb{R}^l$ allocated to consumer i , with $i = 1, 2, \dots, m$.

From now on, we make standard assumptions on the utility functions that ensure us to have a nice equilibrium and optimality theory (see, for example, [1]). These utility functions are smooth, monotonic; the indifference surfaces are bounded from below, the Gaussian curvature of any indifference surface is everywhere different from zero. We then have:

THEOREM 1. *In the finite pure-exchange case, if the total resources r do not depend on the states of nature (i.e., $r(v_1) = r(v_2) = \dots = r(v_n)$) and if the allocation $x = (x_1, x_2, \dots, x_m)$ is Pareto efficient, then the commodity bundle allocated to every consumer is independent of the states of nature (i.e., $x_i(v_1) = x_i(v_2) = \dots = x_i(v_n)$ for $i = 1, 2, \dots, m$). Furthermore, the price vector p supporting the Pareto efficient allocation x is also independent of the states of nature.*

Note that in the statement of the theorem, the initial endowments figure only after having been summed up into the vector of total resources.

Proof. *Step 1.* Consider the case where every event is equiprobable, i.e., $\pi = (1/n, 1/n, \dots, 1/n)$. Let $x = (x_1, x_2, \dots, x_m)$ be the Pareto efficient allocation. Consider the allocation $x^\sigma = (x_1^\sigma, x_2^\sigma, \dots, x_m^\sigma)$, where σ is any permutation of N . It results from $x_i = r$ that we have $\sum x_i^\sigma = r^\sigma = r$ and that x^σ is feasible. We deduce from axiom 2 $u_i(x_i^\sigma) = u_i(x_i)$ for every i . This implies $x^\sigma = x$ for, otherwise, the allocation $x + x^\sigma/2$ would be strictly Pareto superior to x , a contradiction with the Pareto optimality of x . It results from the relation $x^\sigma = x$ for any permutation σ of N that x does not depend on the states of nature, which proves the first part of the theorem in the case of equiprobable events.

To prove the second part, it is just sufficient to show that, taking i arbitrarily, the vector $\text{grad } u_i(x_i)$ does not depend on the states of nature when $x = (x_1, x_2, \dots, x_m)$ is Pareto efficient. From the definition of the gradient vector, one readily checks the relation

$$[\text{grad } u_i(x_i)]^\sigma = \text{grad } u_i(x_i^\sigma)$$

for any permutation σ of N . We know from the first part that x_i^σ is equal to x_i , so that we have

$$[\text{grad } u_i(x_i)]^\sigma = \text{grad } u_i(x_i)$$

for any r . This ends the proof in the equiprobable case.

Step 2. Now, we assume that the probability π_i of event v_i is a rational number for every i . Using the construction mentioned in relationship with Axiom 3, we can define a set of equiprobable events N' such that N defines a partition of N' . The embedding defined by axiom 1 enables us to associate with the allocation $x = (x_1, x_2, \dots, x_m)$ corresponding to the economy associated with N an allocation still denoted $x = (x_1, x_2, \dots, x_m)$ in the economy associated with N' ; let us show that if x is Pareto optimal in the original economy associated with N , then it is also Pareto optimal in the economy associated with N' . Suppose it is not Pareto optimal. Then, there exists an element $(x'_1, x'_2, \dots, x'_m) \in (\mathbb{R}^l)^{N'} \times \dots \times (\mathbb{R}^l)^{N'}$ which Pareto dominates (x_1, x_2, \dots, x_m) naturally embedded in $(\mathbb{R}^l)^{N'} \times \dots \times (\mathbb{R}^l)^{N'}$. Let G

be the set of permutation σ of N' which leaves the partition N invariant (i.e., $\sigma v \sim v$ for any $v \in N'$ and $\sigma \in G$). Clearly, G is a subgroup of the group of permutation of N' not reduced to unity if N contains events with different probabilities. We have $x^\sigma = x$ for every $\sigma \in G$. From axiom 2, we have $u_i(x_i^\sigma) = u_i(x_i)$. Assume there is $\sigma \in G$ such that $x_i^\sigma \neq x_i$ for some i . Then, the allocation $1/\#G \sum_{\sigma \in G} x^\sigma$ is strictly Pareto superior to x' and is invariant by any $\sigma \in G$, so that it defines in fact an element of $(\mathbb{R}^I)^N \times \dots \times (\mathbb{R}^I)^N$ (through the natural embedding) which is Pareto superior to x in the economy associated with N , a contradiction. Therefore, the allocation x viewed as an allocation for the economy associated with N' is Pareto optimal in this economy. Now, we can apply step 1: x_i is invariant for any permutation of N' , which implies $x_i(v_1) = x_i(v_2) = \dots = x_i(v_n)$ for every i .

To prove that p does not depend on the states of nature, we consider $\text{grad } u_i(x_i)$. The invariance property results from the commutative diagram of Axiom 1 which relates $\text{grad } u_i(x_i)$ for the two sets N and N' , and from step 1.

Step 3. We now assume that π contains irrational coordinates. Let U be a compact neighborhood of \bar{u} in P such that the utility functions are defined for $\pi^1 \in U$. Let x be an allocation (not yet a Pareto optimum). Consider the set $K(x, \pi) = \{y \in (\mathbb{R}^I)^N \times \dots \times (\mathbb{R}^I)^N \mid u_i(y_i) \geq u_i(x_i), i = 1, 2, \dots, m\}$, where the utility functions are associated with π . Let $\partial(x, \pi)$ be the diameter of $K(x, \pi)$. Note that $K(x, \pi)$ being convex compact $\partial(x, \pi)$ is finite. Furthermore, one sees readily that $\partial(x, \pi)$ is a continuous function of x and π ; the preferences being strictly convex, $\partial(x, \pi) = 0$ is equivalent to x Pareto optimal. If x is not Pareto optimal, i.e., $\partial(x, \pi) > 0$, then $K(x, \pi)$ contains an allocation y which is Pareto efficient (and Pareto superior to x). Let us apply these properties to the study of the Pareto efficient allocations.

Let x be Pareto efficient for (N, π) ; from $\partial(x, \pi) = 0$, we can choose $\pi^k \in U$ with rational coordinates close enough to π so that $\partial(x, \pi^k)$ is $< 1/R$. In $K(x, \pi^k)$, there exists at least a Pareto efficient allocation denoted x^k , i.e., $\partial(x^k, \pi^k) = 0$. We have $d(x, x^k) < \partial(x, \pi^k) < 1/R$. Therefore, the sequence x^k tends to x . Since x^k is Pareto efficient (for π^k given), we can apply step 2 to x^k , i.e., we have $x^k(v_1) = x^k(v_2) = \dots = x^k(v_m)$. At the limit, we obtain

$$x(v_1) = x(v_2) = \dots = x(v_m).$$

The second part of the theorem results from a limit argument applied to $\text{grad } u_i(x_i)|_\pi = \lim_{k \rightarrow \infty} \text{grad } u_i(x_i^k)|_{x^k}$, the continuity resulting from Axiom 3.

Q.E.D.

COROLLARY. *If the total resources do not depend on the states of nature and if there is extrinsic uncertainty, then the equilibrium prices and the*

competitive allocations do not depend on the states of nature in a finite general equilibrium model.

This is a straightforward consequence of the theorem once one remarks that a competitive allocation is necessarily Pareto efficient and the corresponding equilibrium price vector is the supporting price vector.

4. THE OVERLAPPING-GENERATIONS MODEL AND EXTRINSIC UNCERTAINTY

THEOREM 2. *As in the finite model, Pareto efficient allocations and the supporting price vector are independent of the states of nature.*

In fact, Theorem 2 is exactly Theorem 1 reformulated for the infinite model. It is straightforward that the proof of Theorem 1 works exactly the same in the infinite model. Therefore, the differences that can be established between the two models will not result from different properties of Pareto efficient allocations (since they are identical) but from differences in the relationship between competitive equilibria and Pareto efficient allocations.

We have:

COROLLARY 2. *If the total resources do not depend on the states of nature and if there is extrinsic uncertainty, then the equilibrium prices such that the resulting competitive allocations are Pareto efficient do not depend on the states of nature.*

Of course, the associated competitive allocation does not depend on the states of nature.

In this step, it is important to remark that not all competitive allocations are Pareto optimal. Some may be only weakly Pareto optimal in the sense of [2] and Theorem 2 is not true for W.P.O. allocations. It is an as yet unpublished result of Cass and Shell that W.P.O. allocations may indeed depend on the states of nature.

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Problems of Fair Division and the Egalitarian Solution*

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The problem of fair division is considered in situations where the number of agents among whom the division is to take place may vary. The approach is axiomatic. Along with four familiar axioms, *Weak Pareto-optimality*, *Symmetry*, *Independence of Irrelevant Alternatives*, and *Continuity*, a new axiom, named *Monotonicity with Respect to Changes in the Number of Agents*, is imposed, expressing a certain form of solidarity among the agents as their number varies but the resources at their disposal remain fixed. The only solution to satisfy these axioms is the *Egalitarian solution*, which selects the only feasible alternative that yields equal utilities to all agents and is undominated by any other feasible alternative. *Journal of Economic Literature* Classification Numbers: 024, 025.

1. INTRODUCTION

The traditional description of the problem of fair division involves a group of agents and a list of items that have to be distributed among the agents. An example is the distribution of an inheritance among several heirs. The preferences of the agents over the possible distributions differ and the issue arises as to how to reach an equitable compromise. Somewhat more abstractly, one often simply defines a *division problem* by specifying the utilities assigned by the agents to the "alternatives" open to them, without engaging in a precise physical description of these alternatives. A certain class of division problems, given by the properties they must all have (e.g., convexity, compactness,...), is considered and the search is for division principles or *solutions*, according to which one could perform, for each problem in that class, an optimal division. The axiomatic approach, followed here, consists in formulating desiderata, or axioms, on how this division should be carried out and in checking whether the axioms are compatible.

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