JOURNAL OF ECONOMIC THEORY 56, 206-217 (1992)

Notes, Comments, and Letters to the Editor

The Connectedness of the Set of Equilibrium Money Prices Depends on the Choice of the Numeraire*

ROD GARRATT

Department of Economics, University of California, Santa Barbara, California 93106

Received July 30, 1990; revised May 19, 1991

Peck [*J. Econ. Theory* 43 (1987), 348-354] gives an example of a static general equilibrium economy with lump-sum taxes and transfers in which the set of equilibrium money prices is not connected. Here an example is presented, based on that of Peck, in which for a given tax policy the connectedness of the set of equilibrium money prices depends on the choice of the numeraire. *Journal of Economic Literature* Classification Numbers: D51, E49, H20.

1. INTRODUCTION

For the static general equilibrium economy with lump-sum taxes and transfers denominated in money, Balasko and Shell [3, pp. 15-16] show that the set of equilibrium money prices is the intersection of a ray and an arc-connected set and that it can be not connected.[1] That is, for a given tax policy as the price of money is increased eventually some taxed consumer becomes "bankrupt" but it is possible that as the price of money is further increased eventually all consumers become solvent again and competitive monetary equilibrium is restored. Using the framework of Balasko and Shell [3], Peck [7] provides a concrete example in which the set of equilibrium money prices is not connected when commodity 1 is the numeraire.[2]

In this note, I construct an example based on that of Peck [7] with the property that, for the same tax policy, the set of money prices is a (connected) interval when commodity 2 is the numeraire, even though the set of money prices is not connected when commodity 1 is the numeraire. The preferences in this example are strictly increasing and strictly convex. Each indifference set is a smooth surface and the closure of each indifference surface is contained in the strictly positive orthant. Hence, in a static, monetary, exchange economy which satisfies the regularity assumptions of Balasko and Shell [3], the connectedness of the set of equilibrium money prices can be affected by the choice of the numeraire.

2. THE MODEL

I employ the notation of Balasko and Shell [3], with refinements necessarry for identifying the numeraire choice. The example is for the case of three consumers and two commodities. Let $\omega = (\omega_1, \omega_2, \omega_3) \in (\mathbb{R}^2_{++})^3$ represent the initial endowments of the economy where $\omega_h = (\omega_h^1, \omega_h^2)$ represents the (strictly positive) initial endowments of consumer h ($h = 1, 2, 3$) of commodities 1 and 2. Denote commodity prices by the vector $p = (p^1, p^2) \in \mathbb{R}^2_{++}$ where p^i represents the price of commodity i. A tax policy is equivalent to a vector $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ where $|\tau_h|$ represents the money tax paid by consumer h if τ_h is positive or money transfer (i.e., negative money tax) received by consumer h if τ_h is negative. Let $p^{m,(i)} \in \mathbb{R}_+$ be the price of money when commodity i is the numeraire, $i = 1, 2$, and let the set of equilibrium money prices for given endowments, ω, and tax policy, τ, be denoted by $P^{m,(i)}(\omega, \tau) \subset \mathbb{R}_+$.

Depending on the choice of the numeraire, each tax policy can be represented by a reallocation of endowments. That is, a money tax on consumer h equal to τ_h is equivalent to a tax of $p^{m,(i)}\tau_h$ units of (the numeraire) commodity i. It will be useful when calculating the set of equilibrium money prices corresponding to the given tax policy to define the associated endowment vector for each choice of the numeraire. Let $\tilde{\omega}_h^{(1)} = (\omega_h^1 - p^{m,(1)}\tau_h, \omega_h^2)$ be the associated endowment vector of consumer h in the case where commodity 1 is the numeraire and let $\tilde{\omega}_h^{(2)} = (\omega_h^1, \omega_h^2 - p^{m,(2)}\tau_h)$ be the associated endowment vector of consumer h in the case where commodity 2 is the numeraire.

The associated endowment vector of a taxed consumer will have negative

* This paper is part of my doctoral dissertation which was written while I was at Cornell University. I am grateful to Karl Shell for his encouragement and helpful comments. I also thank an anonymous referee. Research support from National Science Foundation Grant SES-9012780, the Thorne fund, and the Center for Analytic Economics at Cornell University is gratefully acknowledged.

[1] For further results also see Balasko and Shell [4].

[2] Balasko and Shell [2] and Peck [8] do the same thing for the (infinite) perfect-foresight overlapping-generations model.

0022-0531/92 $3.00

components for some prices of money. However, given the specification of preferences in this example, it is required that in equilibrium the consumption of each consumer (denoted by the vector $x_h = (x_h^1, x_h^2)$ where x_h^i represents the consumption of commodity i by consumer h) lie in the strictly positive orthant.

3. The Example

As in Peck [7], the initial endowments are specified by $\omega_1 = (4, 8)$, $\omega_2 = (20, 4)$, $\omega_3 = (4, 8)$ and the tax policy is given by $\tau = (20, 0, -20)$. Preferences for each consumer are completely described as follows[3]: For consumer 1 (see Fig. 1), the preference map is built from three functions, $u_1^k(x_1^1): \mathbb{R}_{++} \to \mathbb{R}_{++}$, $k = 1, 2, 3$, which are assumed to be smooth, strictly decreasing, and strictly convex and satisfy $u_1^1(x_1^1) < u_1^2(x_1^1) < u_1^3(x_1^1)$ for $x_1^1 > 0$. Furthermore, it is assumed that the function $u_1^k(x_1^1)$ has the limits

$$\lim_{x_1^1 \to \infty} u_1^k(x_1^1) = 0 \qquad \text{and} \qquad \lim_{x_1^1 \to 0} u_1^k(x_1^1) = \infty,$$

for each k.[4] Using the functions $u_1^k(x_1^1)$ the curves u_1^k are defined as the points $(x_1^1, u_1^k(x_1^1))$ as x_1^1 varies over $\mathbb{R}_{++}$. The curves have slopes of -2 at the point $(2, 4)$ on u_1^1, $-\frac{1}{4}$ at the point $(4, 3)$ on u_1^2, and -1 at the point $(8, 4)$ on u_1^3. Below u_1^1 use radial projections of the curve u_1^1 towards the origin. This may be done in a way which ensures that the preferences of consumer 1 are homothetic to the southwest of u_1^1 (in particular the slopes of the curves will be equal to -2 everywhere on the ray joining $(0, 0)$ to $(2, 4)$).[5] Indifference curves between u_1^1 and u_1^2 and between u_1^2 and u_1^3 are in each case found using convex combinations of the two curves along rays from the origin. Above u_1^3 indifference curves are formed using radial projections of u_1^3 away from the origin.

For consumer 2 (see Fig. 2), the preference map is built from four functions, $u_2^k(x_2^1): \mathbb{R}_{++} \to \mathbb{R}_{++}$, $k = 1, 2, 3, 4$, which are assumed to be smooth, strictly decreasing, and strictly convex and satisfy $u_2^1(x_2^1) <$

[3] The technique used in this example to construct the preference map of each consumer was also used by Peck [7]. Peck attributes his inspiration for this method of filling in all the indifference curves from a few specified ones to Cass, Okuno, and Zilcha [6]. A detailed explanation of these constructions, including precisely what is meant by the various types of "convex combinations" or "radial projections" which are used here, is found in the Appendix of the Cass-Okuno-Zilcha paper [6, pp. 71–75].

[4] In order to define a preference map which has the property that the closure of each indifference surface in $\mathbb{R}^2$ is contained in $\mathbb{R}^2_{++}$, the limits of the functions, $u_h^k(x_h^1)$, used here differ from those specified by Cass, Okuno, and Zilcha.

[5] See Cass, Okuno, and Zilcha [6, pp. 76–77].

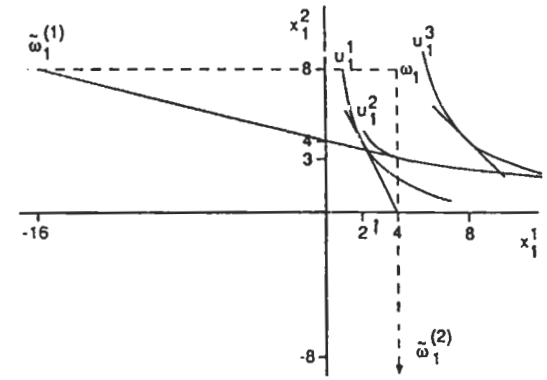

Figure 1

$u_2^2(x_2^1) < u_2^3(x_2^1) < u_2^4(x_2^1)$ for $x_2^1 > 0$. Furthermore, it is assumed that the function $u_2^k(x_2^1)$ has the limits

$$\lim_{x_2^1 \to \infty} u_2^k(x_2^1) = 0 \qquad \text{and} \qquad \lim_{x_2^1 \to 0} u_2^k(x_2^1) = \infty,$$

for each k. Using the functions $u_2^k(x_2^1)$ the curves u_2^k are defined as the points $(x_2^1, u_2^k(x_2^1))$ as x_2^1 varies over $\mathbb{R}_{++}$. The curves have slopes of $-\frac{1}{4}$ at the point $(4, 8)$ on u_2^1, $-\frac{1}{3}$ at the point $(16, 5\frac{1}{3})$ on u_2^2, -4 at the point

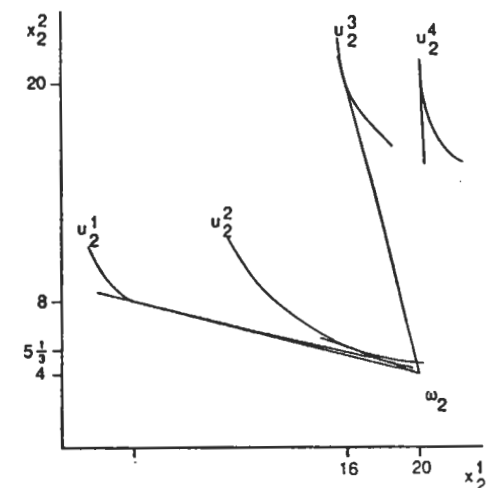

Figure 2

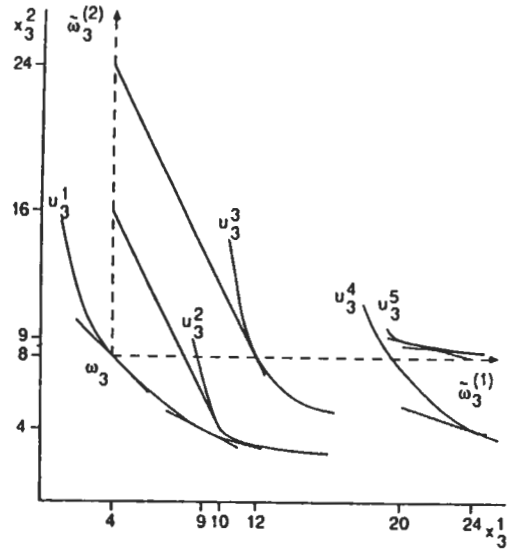

FIGURE 3

(16, 20) on u_2^3, and -20 at the point (20, 20) on u_2^4. Below u_2^1 the indifference curves are filled in using convex combinations of points on u_2^1 and the point (0, 0). Between u_2^1 and u_2^2 convex combinations of these two curves along rays from the origin are used. The indifference curves between u_2^2 and u_2^3 are found by forming convex combinations of the two curves along vertical lines. As a result of this construction, the offer curve between u_2^2 and u_2^3 is a vertical segment given the endowment point (20, 4).[6] The indifference curves between u_2^3 and u_2^4 are found by forming convex combinations of the two curves along horizontal lines. Radial projections of u_2^4 are used above u_2^4.

For consumer 3 (see Fig. 3), the preference map is defined from five functions, $u_3^k(x_3^1)$: $\mathbf{R}_{++} \to \mathbf{R}_{++}$, $k = 1, 2, 3, 4, 5$, which are assumed to be smooth, strictly decreasing, and strictly convex and satisfy $u_3^1(x_3^1) < u_3^2(x_3^1) < u_3^3(x_3^1) < u_3^4(x_3^1) < u_3^5(x_3^1)$ for $x_3^1 > 0$. Furthermore, it is assumed that the function $u_3^k(x_3^1)$ has the limits

$$\lim_{x_3^1 \to \infty} u_3^k(x_3^1) = 0 \qquad \text{and} \qquad \lim_{x_3^1 \to 0} u_3^k(x_3^1) = \infty,$$

for each k. Using the functions $u_3^k(x_3^1)$ the curves u_3^k are defined as the points $(x_3^1, u_3^k(x_3^1))$ as x_3^1 varies over $\mathbf{R}_{++}$. The curve u_3^1 has a slope of -1

[6] It is shown that this method of construction leads to a vertical offer curve relative to a fixed endowment point in the Appendix of the Cass, Okuno, and Zilcha [6, pp. 74–75].

at the point (4, 8) and $-\frac{3}{4}$ at the point (9, 4). The curves u_3^2 and u_3^3 each have slopes of -2 at the points (10, 4) and (12, 8), respectively. The curve u_3^4 has a slope of $-\frac{1}{3}$ at the point (24, 4). The curve u_3^5 has a slope of $-\frac{1}{4}$ at the point (20, 9) and $-\frac{1}{12}$ at the point $(24, 8\frac{1}{2})$. Below u_3^1 convex combinations of the points of u_3^1 and the point (0, 0) are used. Indifference curves between u_3^1 and u_3^2 are formed using convex combinations of the two curves along horizontal lines. Indifference curves between u_3^2 and u_3^3 are formed using convex combinations of the two curves along lines parallel to the ray joining (10, 4) to (12, 8). This ensures that the indifference curves between u_3^2 and u_3^3 have slopes of -2 everywhere on the ray joining (10, 4) to (12, 8). (See the Appendix.) Indifference curves between u_3^3 and u_3^4 are formed using convex combinations of the two curves along rays from the origin. Between u_3^4 and u_3^5 convex combinations of the two curves along vertical lines are used. Above u_3^5, radial projections of u_3^5 are used.

Note that the each of the preference maps which have just been described corresponds to a utility function, U_h, which is continuous, strictly increasing, and strictly quasi-concave.[7]

CLAIM 1. *$P^{m,(1)}(\omega, \tau)$, the set of equilibrium money prices when commodity 1 is the numeraire, is not a connected set.*

Proof. Since the numeraire is commodity 1, the effect on the economy of altering the price of money (and/or altering the tax) can be represented by taxes and transfers of commodity 1 which are reflected in the associated endowment vectors $\tilde{\omega}_h^{(1)}$.[8] When $p^{m,(1)} = 0$ we have an equilibrium based on the initial endowment with prices $p = (1, 1)$ and allocations $x_1 = (8, 4)$, $x_2 = (16, 8)$, $x_3 = (4, 8)$. When $p^{m,(1)} = 1$ the associated endowment is $\tilde{\omega}_1^{(1)} = (-16, 8)$, $\tilde{\omega}_2^{(1)} = (20, 4)$, $\tilde{\omega}_3^{(1)} = (24, 8)$ and we have an equilibrium for the associated nonmonetary economy with prices $p = (1, 4)$ and allocations $x_1 = (4, 3)$, $x_2 = (4, 8)$, $x_3 = (20, 9)$. Hence there exists a monetary equilibrium in the original economy at $p^{m,(1)} = 1$ with commodity prices $p = (1, 4)$. When $p^{m,(1)} = \frac{2}{5}$ the associated endowment is $\tilde{\omega}_1^{(1)} = (-4, 8)$, $\tilde{\omega}_2^{(1)} = (20, 4)$, $\tilde{\omega}_3^{(1)} = (12, 8)$. If $p^2 \leqslant \frac{1}{2}$ then consumer 1 will have zero or

[7] The proof of this assertion is found in the Appendix of Cass, Okuno, and Zilcha [6, p. 74] for the case where the function $u_h^k(x_h^1)$ (in the notation of this paper) is allowed to intersect the horizontal and the vertical axis as x_h^1 varies over $\mathbf{R}_+$. In the Cass–Okuno–Zilcha paper the utility level assigned to each indifference curve corresponds to the intersection of each indifference curve with the vertical axis. However, it is not essential to their argument that utilities be assigned this way. The Cass–Okuno–Zilcha argument can be repeated in the framework of this example by defining the utility level assigned to each indifference curve according to each indifference curve's intersection with any vertical line through the strictly positive portion of the horizontal axis.

[8] This result is proven for the T period overlapping-generations model in Balasko and Shell [5, pp. 136–138].

negative wealth and thus can not consume inside the consumption set. If $\frac{1}{2} < p^2 \leqslant 3$ then $x_2^1 = 16$ and $x_3^1 > 12$ so there is excess demand for commodity 1. If $p^2 > 3$ then $x_1^1 > 4$. Now examine the demand for commodity 1 by consumer 3. Consider a ray from the point (24, 4) with slope $-\frac{1}{3}$ and a ray from the point $(24, 8\frac{1}{2})$ with slope $-\frac{1}{12}$. These rays intersect at the point (6, 10). Since the indifference curves between u_3^4 and u_3^5 are constructed using convex combinations of the two curves along vertical lines, the offer curve relative to consumer 3's hypothetical endowment point (6, 10) is the vertical segment between the points (24, 4) and $(24, 8\frac{1}{2})$. However, for each $p^2 > 3$ the budget lines relative to the point (12, 8) (which lies on the ray from the point (24, 4) with slope $-\frac{1}{3}$) all have a slope strictly greater, as they pass through the vertical segment between the points (24, 4) and $(24, 8\frac{1}{2})$, than the slopes of the budget lines relative to the point (6, 10). Given convexity of the indifference curves it follows that for $p^2 > 3$, $x_3^1 > 24$. Thus for $p^2 > 3$, $x_1^1 > 4$ and $x_3^1 > 24$ so there is excess demand for commodity 1. Thus there is no equilibrium for the associated nonmonetary economy at $p^{m,(1)} = \frac{2}{3}$ and therefore there is no monetary equilibrium at $p^{m,(1)} = \frac{2}{3}$. Therefore the set $P^{m,(1)}(\omega, \tau)$ is not connected. ∎

CLAIM 2. *$P^{m,(2)}(\omega, \tau)$, the set of equilibrium money prices when commodity 2 is the numeraire, is a connected set.*

Proof. Since the numeraire is commodity 2 the effect on the economy of altering the price of money (and/or altering the tax) can be represented by taxes and transfers of commodity 2 which are reflected in the associated endowment vectors $\tilde{\omega}_h^{(2)}$. For $\tilde{\omega}^{(2)} = (\tilde{\omega}_1^{(2)}, \tilde{\omega}_2^{(2)}, \tilde{\omega}_3^{(2)}) \in \mathbb{R}^3_{++}$ there exists an equilibrium for the associated nonmonetary economy.[9] Hence there exists a monetary equilibrium in the original economy at each $p^{m,(2)} \in [0, \frac{2}{3})$.

Now consider $p^{m,(2)} \in [\frac{2}{3}, \frac{4}{3})$. As $p^{m,(2)}$ is increased from $\frac{2}{3}$ towards $\frac{4}{3}$ the associated endowment vector, $\tilde{\omega}^{(2)}$, varies from ((4, 0), (20, 4), (4, 16)) towards ((4, −8), (20, 4), (4, 24)) and there exists a nonmonetary competitive equilibrium for each associated endowment in this sequence where $p = (2, 1)$, $x_2 = (16, 12)$, and x_1 decreases linearly from (2, 4) towards (0, 0) as x_3 increases linearly from (10, 4) towards (12, 8). Hence there exists a monetary equilibrium in the original economy at each $p^{m,(2)} \in [\frac{2}{3}, \frac{4}{3})$.

For $p^{m,(2)} \in [\frac{4}{3}, 1\frac{1}{3})$ there does not exist a monetary competitive equilibrium. Consider an arbitrary $p^{m,(2)} \in [\frac{4}{3}, 1\frac{1}{3})$ and let $\tilde{\omega}^{(2)}$ be the associated endowment where $\tilde{\omega}_1^{(2)} \in \{(\tilde{\omega}_1^{1,(2)}, \tilde{\omega}_1^{2,(2)}) \mid \tilde{\omega}_1^{1,(2)} = 4, \ -16 < \tilde{\omega}_1^{2,(2)} \leqslant -8\}$, $\tilde{\omega}_2^{(2)} = (20, 4)$, and $\tilde{\omega}_3^{(2)} \in \{(\tilde{\omega}_3^{1,(2)}, \tilde{\omega}_3^{2,(2)}) \mid \tilde{\omega}_3^{1,(2)} = 4, \ 24 \leqslant \tilde{\omega}_3^{2,(2)} < 32\}$. For $p^1 \leqslant -\tilde{\omega}_1^{2,(2)}/4$ consumer 1 has zero or negative wealth. If $-\tilde{\omega}_1^{2,(2)}/4 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/6$ then $x_2^2 + x_3^2 > (4p^1 + 4) + (48 - 8p^1)/(2 + p^1) > 20$ so there is excess demand for commodity 2.

[9] See Arrow and Hahn [1, pp. 107–128].

Consider prices such that $(\tilde{\omega}_3^{2,(2)} - 4)/6 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/5$. If p^1 is within this range of prices and $p^1 \leqslant 4$ then $x_2^2 = 4p^1 + 4 > 17\frac{1}{3}$ whereas if p^1 is within this range of prices and $p^1 > 4$ then $x_2^2 > 20$. To see the latter result, recall that the indifference curves between u_2^3 and u_2^4 are formed using convex combinations of the two curves along horizontal lines. As a result the offer curve relative to consumer 2's hypothetical endowment point (21, 0) is the horizontal segment between the points (16, 20) and (20, 20). However for each p^1 such that p^1 is within the range of prices $(\tilde{\omega}_3^{2,(2)} - 4)/6 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/5$ and $p^1 > 4$ the budget lines relative to the point (20, 4) (which lies on the ray from the point (16, 20) with slope −4) all have a slope strictly less, as they pass through the horizontal segment between the points (16, 20) and (20, 20), than the slopes of the budget lines relative to the point (21, 0). Given convexity of the indifference curves it follows that $x_2^2 > 20$. Recall that the indifference curves between u_3^1 and u_3^2 are formed using convex combinations of the two curves along horizontal lines. As a result the offer curve relative to consumer 3's hypothetical endowment point $(10\frac{3}{5}, 2\frac{4}{5})$ is a horizontal ray connecting (9, 4) to (10, 4). This implies that the slopes of the indifference curves between u_3^1 and u_3^2 as they pass through this horizontal segment vary from −2 to $-\frac{3}{4}$. However for each $(\tilde{\omega}_3^{2,(2)} - 4)/6 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/5$ the budget lines originating from consumer 3's associated endowment point $(4, \tilde{\omega}_3^{2,(2)})$ pass through the horizontal segment between (9, 4) and (10, 4) with slopes less than $-3\frac{1}{3}$. Given convexity of the indifference curves it follows that for $(\tilde{\omega}_3^{2,(2)} - 4)/6 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/5$, $x_3^2 > 4$. Thus for $(\tilde{\omega}_3^{2,(2)} - 4)/6 < p^1 \leqslant (\tilde{\omega}_3^{2,(2)} - 4)/5$, $x_2^2 > 17\frac{1}{3}$ and $x_3^2 > 4$ so there is excess demand for commodity 2. Consider $p^1 > (\tilde{\omega}_3^{2,(2)} - 4)/5$. Again note that the offer curve relative to consumer 2's hypothetical endowment point (21, 0) is the horizontal segment between the points (16, 20) and (20, 20). However, for each $p^1 > (\omega_3^{2,(2)} - 4)/5$ the budget lines relative to the point (20, 4) (which lies on the ray from the point (16, 20) with slope −4) all have a slope strictly less, as they pass through the horizontal segment between the points (16, 20) and (20, 20), than the slopes of the budget lines relative to the point (21, 0). Given convexity of the indifference curves it follows that for $p^1 > (\tilde{\omega}_3^{2,(2)} - 4)/5$, $x_2^2 > 20$ so there is excess demand for commodity 2. Thus there is no equilibrium for the associated nonmonetary economy for $p^{m,(2)} \in [\frac{4}{3}, 1\frac{1}{3})$ and therefore there is no monetary equilibrium for $p^{m,(2)} \in [\frac{4}{3}, 1\frac{1}{3})$.

Finally suppose $p^{m,(2)} \in [1\frac{1}{3}, \infty)$. If $p^1 \leqslant 4$ then consumer 1 has zero or negative wealth. If $p^1 > 4$ then $x_2^2 > 20$ so there is excess demand for commodity 2. Thus there is no equilibrium for the associated nonmonetary economy for $p^{m,(2)} \in [1\frac{1}{3}, \infty)$ and therefore there is no monetary equilibrium for $p^{m,(2)} \in [1\frac{1}{3}, \infty)$. Thus the set of equilibrium money prices is $P^{m,(2)}(\omega, \tau) = [0, \frac{4}{3})$. ∎

The results are summarized in the following proposition.

PROPOSITION 1. *In some static, monetary, exchange economies the connectedness of the set of equilibrium money prices depends on the choice of the numeraire.*

Proof. Immediate from Claims 1 and 2. ∎

4. CONCLUDING REMARKS

It has been shown that for an example of the static, monetary, exchange economy with regular preferences, the connectedness of the set of equilibrium money prices depends on the numeraire choice. The same example can be reinterpreted in terms of a nonmonetary exchange economy in which endowments may lie outside the strictly positive (or positive) orthant and therefore outside the consumers' consumption sets. Consider the mapping which takes endowments into equilibrium prices. I have given an example in which the projection of the equilibrium set down to the endowment space has a "hole" in the direction of the first endowment, as we move away from the initial endowment, but does not have a hole in the direction of the second endowment.

My analysis is based on two methods of normalization: $p^1 = 1$ and $p^2 = 1$. The more general normalization would be $\phi(p^1, p^2) = 1$ where $\phi(\cdot)$ is a nondecreasing function of (p^1, p^2) and is increasing in either p^1 or p^2 (or both). Given the economy and the tax policy, can one always find a normalization $\phi(\cdot)$ such that the resulting set of equilibrium money prices is connected? That is, is it impossible for the above-mentioned projection to have a piece outside the strictly positive orthant that is not connected to the piece contained in the strictly positive orthant? This question remains to be solved.

The question of whether the connectedness of the set of equilibrium money prices depends on the choice of the numeraire in the (infinite) overlapping-generations model of Balasko and Shell [2] is also left to future research; cf. Peck [8].

APPENDIX

The construction of indifference curves beteen u_3^2 and u_3^3 which each have a slope of -2 along the ray connecting the points (10, 4) and (12, 8) proceeds as follows.[10] The curves between u_3^2 and u_3^3 have been defined relative to a Cartesian plane with x_3^1 on the horizontal axis and x_3^2 on the vertical axis. Relative to this set of points, define a new pair of axes with z_3^1 and z_3^2 representing points on the horizontal axis and the vertical axis, respectively, by shifting the intersection of the x_3^1 axis and the x_3^2 axis from the point (0, 0) to the point (8, 0) and rotating the plane so that the z_3^1 axis in the newly defined Cartesian plane has a slope of $-\frac{1}{2}$ relative to the original plane (See Fig. 4.). Relative to the new coordinates the curves u_3^2 and u_3^3 each have a slope of $-\frac{3}{4}$ at the points $(0, \sqrt{20})$ and $(0, \sqrt{80})$, respectively. The desired construction may now be performed with respect to the newly defined coordinates.

[10] The technique used here is based on that used by Cass, Okuno, and Zilcha [6, pp. 71–75] to construct an example of a utility function which yields a vertical segment on the offer curve.

It must be shown that curves may be constructed between u_3^2 and u_3^3 which each have a slope of $-\frac{3}{4}$ along the newly defined z_3^2 axis. The curves u_3^2 and u_3^3 have the following relationships to the budget constraints $\sqrt{20} - \frac{3}{4}z_3^1$ and $\sqrt{80} - \frac{3}{4}z_3^1$ (as defined using the new coordinates over the appropriate range) respectively:

$$u_3^2(z_3^1) \begin{Bmatrix} > \\ = \end{Bmatrix} \sqrt{20} - \tfrac{3}{4}z_3^1 \quad \text{as} \quad z_3^1 \begin{Bmatrix} \neq \\ = \end{Bmatrix} 0$$

and

$$u_3^3(z_3^1) \begin{Bmatrix} > \\ = \end{Bmatrix} \sqrt{80} - \tfrac{3}{4}z_3^1 \quad \text{as} \quad z_3^1 \begin{Bmatrix} \neq \\ = \end{Bmatrix} 0.$$

Let

$$\alpha = \frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}},$$

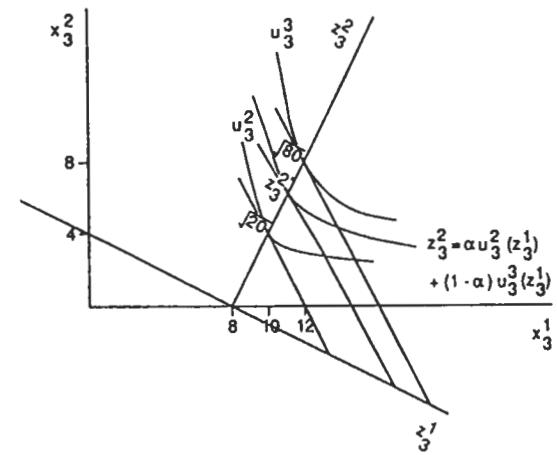

FIGURE 4

where $\sqrt{20} \leqslant z_3^{2\prime} \leqslant \sqrt{80}$. The indifference curves between u_3^2 and u_3^3 are defined as convex combinations of $(z_3^1, u_3^2(z_3^1))$ and $(z_3^1, u_3^3(z_3^1))$ using the weights α and $1-\alpha$, respectively. That is, the indifference curves beteen u_3^2 and u_3^3 are defined by the equations

$$z_3^2 = \frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}}\, u_3^2(z_3^1) + \left(1 - \left(\frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}}\right)\right) u_3^3(z_3^1)$$

for $z_3^1 \in \mathbb{R}$ and $\sqrt{20} \leqslant z_3^{2\prime} \leqslant \sqrt{80}$. It remains to verify that each of these indifference curves so constructed will have a slope of $-\frac{3}{4}$ at the points $(0, z_3^{2\prime})$, $\sqrt{20} \leqslant z_3^{2\prime} \leqslant \sqrt{80}$. Note that the budget constraint $z_3^2 = z_3^{2\prime} - \frac{3}{4} z_3^1$ is identical to that obtained by taking the same convex combination of the two budget constraints $\sqrt{20} - \frac{3}{4} z_3^1$ and $\sqrt{80} - \frac{3}{4} z_3^1$. That is,

$$z_3^{2\prime} - \tfrac{3}{4} z_3^1 = \frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}} \left(\sqrt{20} - \tfrac{3}{4} z_3^1\right) + \left(1 - \left(\frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}}\right)\right) \left(\sqrt{80} - \tfrac{3}{4} z_3^1\right).$$

Thus it follows that

$$\frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}}\, u_3^2(z_3^1) + \left(1 - \left(\frac{\sqrt{80} - z_3^{2\prime}}{\sqrt{80} - \sqrt{20}}\right)\right) u_3^3(z_3^1) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} z_3^{2\prime} - \tfrac{3}{4} z_3^1 \quad \text{as} \quad z_3^1 \left\{ \begin{matrix} \neq \\ = \end{matrix} \right\} 0$$

for $\sqrt{20} \leqslant z_3^{2\prime} \leqslant \sqrt{80}$, which is the desired result.

References

1. K. J. Arrow and F. H. Hahn, "General Competitive Analysis," Holden-Day, San Francisco, 1971.
2. Y. Balasko and K. Shell, The overlapping-generations model. II. The case of pure exchange with money, *J. Econ. Theory* 24 (1981), 112–142.
3. Y. Balasko and K. Shell, "Lump-Sum Taxation: The Static Economy," CAREES Working Paper 83-08RR, University of Pennsylvania, Philadelphia, PA, August 1, 1983; and *in* "General Equilibrium and Growth: The Legacy of Lionel McKenzie" (R. Becker, M. Boldrin, R. Jones, and W. Thomson, Eds.), Academic Press, New York, in press.

4. Y. Balasko and K. Shell, On taxation and competitive equilibria, *in* "Optimalité et Structures: Mélanges en Hommage à Edouard Rossier" (G. Ritschard and D. Royer, Eds.), pp. 69–83, Economica, Paris, 1985.
5. Y. Balasko and K. Shell, Lump-sum taxes and transfers: Public debt in the overlapping-generations model, *in* "Equilibrium Analysis: Essays in Honor of Kenneth J. Arrow" (W. P. Heller, R. M. Starr, and D. A. Starrett, Eds.), Vol. II, pp. 121–153, Cambridge Univ. Press, Cambridge, 1986.
6. D. Cass, M. Okuno, and I. Zilcha, The role of money in supporting the Pareto optimality of competitive equilibrium in consumption-loan type models, *J. Econ. Theory* 20 (1979), 41–80.
7. J. Peck, Non-connectedness of the set of equilibrium money prices: The static economy, *J. Econ. Theory* 43 (1987), 348–354.
8. J. Peck, Non-connectedness of the set of equilibrium money prices: The overlapping-generations economy, *J. Econ. Theory* 43 (1987), 355–363.