Non-connectedness of the Set of Equilibrium Money Prices: The Static Economy

JAMES PECK

Department of Managerial Economics and Decision Sciences,
J. L. Kellogg Graduate School of Management,
Northwestern University, Evanston, Illinois 60201
Received December 16, 1986; revised May 15, 1987

A static economy in which nominal taxes and transfers are balanced, under certain conditions, has a set of equilibrium money prices containing a proper interval. Two examples are given in which the entire set must consist of a proper interval. Then, an example is presented in which the set of equilibrium money prices is not connected. While the set contains a proper interval for balanced fiscal policies, the entire set is not in general a single interval. *Journal of Economic Literature Classification Numbers: 021, 022, 023. © 1987 Academic Press, Inc.

Balasko and Shell [1] consider a static exchange economy in which the government can levy taxes and distribute transfers in terms of the monetary unit. The price of money in terms of commodities, determined in the market, takes on any one of a range of values. For each balanced fiscal policy, the set of equilibrium money prices is shown to contain an interval $[0, \bar{p}^m]$, where $\bar{p}^m$ is a positive scalar. Balasko and Shell indicate that the set might consist of two unconnected intervals or some more complicated structure, although in all the specific examples given so far the set of equilibrium money prices is an interval. An example is presented here, satisfying all the regularity assumptions of [1], in which the set of equilibrium money prices is not connected. While the set contains a proper interval for balanced fiscal policies, the entire set is not in general a single interval.

Each consumer in the example has preferences giving rise to a utility function that is strictly monotonic, differentiable, and strictly quasi-concave. Endowments are contained in the strictly positive orthant. We follow the notation of Balasko and Shell [1]. In particular, the vector of commodity prices given by $p = (p^1, p^2, \ldots, p^n)$ has commodity one as numeraire, so $p^1 = 1$.

We start by recalling two examples in which the set of equilibrium money prices, $\mathcal{P}^m(\omega, \tau)$, is a proper interval. In Example 3, a non-degenerate example is presented in which $\mathcal{P}^m(\omega, \tau)$ is not an interval.

**Example 1.** Let there be one commodity, $l = 1$. Consider the tax-transfer policy $\tau = (\tau_1, \ldots, \tau_n)$ which is balanced i.e., $\sum_{h=1}^n \tau_h = 0$, and nontrivial, i.e., $\tau \neq 0$. (Positive components of $\tau$ represent taxes and negative components represent transfers.)

**Claim 1.** The set $\mathcal{P}^m(\omega, \tau)$ is a proper interval.

**Proof of Claim 1.** The maximization problem for consumer $h$ is

$$
\begin{align*}
\text{max} & \quad u_h(x_h) \\
\text{s.t.} & \quad p \cdot x_h \leq p \cdot \omega_h - p^m \tau_h \\
& \quad x_h > 0.
\end{align*}
$$

Because there is one commodity, $p$ is a scalar which can be normalized to one. Monotonicity of the utility function implies

$$
\begin{align*}
x_h &= \omega_h - p^m \tau_h \\
x_h &> 0.
\end{align*}
$$

Summing the budget constraints of the consumers, and using the fact that $\tau$ is balanced, we have

$$
\sum_{h=1}^n x_h = \sum_{h=1}^n \omega_h - p^m \sum_{h=1}^n \tau_h = \sum_{h=1}^n \omega_h.
$$

Therefore, markets always clear for any value of $p^m$. Equilibrium obtains if and only if $p^m$ is consistent with a solution to (1) and (2) for all consumers. It follows that the set of equilibrium money prices is specified by

$$
\mathcal{P}^m(\omega, \tau) = \{ p^m \geq 0 | p^m < \min_{\tau_h > 0} [\omega_h / \tau_h] \},
$$

which is a proper interval. □

**Example 2.** Let there be $l$ commodities but only two consumers, $n = 2$. Consider the tax-transfer policy $\tau = (\tau_1, \tau_2)$, which is balanced and nontrivial.

**Claim 2.** The set $\mathcal{P}^m(\omega, \tau)$ is a proper interval.

* Support from the National Science Foundation under Grant SES-83-08450 is acknowledged. I would like to thank Karl Shell for his advice and encouragement. Remaining errors are my own.
Proof of Claim 2. The set of tax-transfer policies consistent with equilibrium and $p^m = 1$, $F_{bon}$, is called the set of normalized bona fide tax-transfer policies. The set $F_{bon}$ must be contained in the one-dimensional subspace defined by $r_1 + r_2 = 0$, since only balanced policies can be normalized bona fide. By Balasko and Shell [1, Proposition 3.6], we know that $F_{bon}$ is arc-connected (and hence connected) and contains zero in its relative interior. A subset of $R$ consisting of more than one point is connected if and only if it is a proper interval, so $F_{bon}$ is a proper interval. It follows from [1, Proposition 2.5] that $\mathcal{P}''(\omega, \tau)$ is also a proper interval.

Example 3 (in which $\mathcal{P}''(\omega, \tau)$ is not an interval). In view of Example 1 and Example 2, we must have at least two commodities and at least three consumers, so let $l = 2$ and $n = 3$. Also, let endowments be

$$\omega_1 = (\omega_{11}, \omega_{12}) = (4, 8),$$
$$\omega_2 = (\omega_{21}, \omega_{22}) = (20, 4),$$
$$\omega_3 = (\omega_{31}, \omega_{32}) = (4, 8),$$

and let the tax-transfer policy be given by $\tau = (r_1, r_2, r_3) = (20, 0, -20)$.

The preferences of the three consumers are depicted in Figs. 1–3. Figure 1 shows two of consumer 1’s indifference curves, labelled $\bar{u}_1^1$ and $\bar{u}_1^2$. The marginal rate of substitution is 1 at the point (8, 4) and 0.25 at the point (4, 3). To complete the preference map, indifference curves below $\bar{u}_1^2$ are formed from convex combinations of $\bar{u}_1^1$ and (0, 0) along rays from the origin. Form convex combinations of $\bar{u}_1^1$ and $\bar{u}_1^2$ along rays from the origin to construct the indifference curves between $\bar{u}_1^1$ and $\bar{u}_1^2$. Form radial projections of $\bar{u}_1^1$ to get the indifference curves above $\bar{u}_1^1$. This procedure guarantees that preferences satisfy the required regularity conditions.

Figure 2 shows three of consumer 2’s indifference curves, labelled $\bar{u}_2^1$, $\bar{u}_2^2$, and $\bar{u}_2^3$. The marginal rates of substitution at the points (16, 8), (4, 8), and (16, 4) are $\frac{3}{4}$, $\frac{1}{4}$, and $\frac{3}{4}$, respectively. Below $\bar{u}_2^3$, construct the missing indifference curves by taking convex combinations of $\bar{u}_2^2$ and (0, 0) along rays from the origin. Form convex combinations of $\bar{u}_2^1$ and $\bar{u}_2^3$ along rays from the origin between those curves, and form radial projections of $\bar{u}_2^3$ above $\bar{u}_2^1$. Between $\bar{u}_2^1$ and $\bar{u}_2^3$, the indifference curves are found by forming convex combinations of the two curves along vertical lines. Preferences are smooth and convex, and the offer curve between $\bar{u}_2^1$ and $\bar{u}_2^3$ is a vertical segment.1

This follows from the fact that the line tangent to $\bar{u}_2^3$ at $x^1 = 16$ and the line tangent to $\bar{u}_2^1$ at $x^1 = 16$ intersect at the endowment point. Figure 3 shows four of consumer 3’s indifference curves, labelled $\bar{u}_3^1$, $\bar{u}_3^2$, $\bar{u}_3^3$, and $\bar{u}_3^4$. The marginal rates of substitution at the points (4, 8), (20, 9), (17, 4), and (30, 4) are $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{4}{3}$, respectively. To fill in the missing indifference curves below $\bar{u}_3^4$, form convex combinations of $\bar{u}_3^4$ and (0, 0) along rays from the origin. Between $\bar{u}_3^4$ and $\bar{u}_3^3$, form convex combinations of the two curves along rays from the origin. Above $\bar{u}_3^3$, form radial projections of $\bar{u}_3^3$. Between $\bar{u}_3^3$ and $\bar{u}_3^1$, form convex combinations of the two curves along horizontal lines. Between $\bar{u}_3^2$ and $\bar{u}_3^1$, form convex combinations of the two curves along vertical lines. Preferences are smooth and convex, and the offer curve between $\bar{u}_3^1$ and $\bar{u}_3^2$ is a horizontal segment.2

Also, when $p^m = 0.5$ and consumer 3’s “translated” endowment point is (14, 8), he will demand at least 20 units of commodity one whenever we have $p^m > 4$.

For a given value of $p^m$, if we assume that the offer curve is given by $\omega_t = (4 - 20(0.5), 8) = (0, 8)$, then the translated endowment point is $\bar{u}_3^1 = (14, 8)$ associated with $p^m = 0.5$.

---

1 This method of filling in the indifference curves to generate vertical or horizontal segments of the offer curve was inspired by Cass, Okuno, and Zilcha [3].

2 Since different values of $p^m$ lead to different translated endowment points, the offer curve varies with $p^m$. The offer curve between $\bar{u}_3^1$ and $\bar{u}_3^2$ is a horizontal segment for the translated endowment $\omega_t = (14, 8)$ associated with $p^m = 0.5$. 
(-6, 8), and when \( p^m = 1 \) holds we have \( \tilde{\omega}_1 = (4 - 20, 8) = (-16, 8) \). Since \( \tau_2 = 0 \), Mr 2's translated endowment always equals his untranslated endowment, \( \tilde{\omega}_2 = \omega_2 = (20, 4) \). For \( p^m = 0.5 \), Mr. 3's translated endowment point is \( \tilde{\omega}_3 = (4 + 20(0.5), 8) = (14, 8) \), and for \( p^m = 1 \) we have \( \tilde{\omega}_3 = (4 + 20, 8) = (24, 8) \). The translated endowments are not all in the positive orthant, so it does not necessarily follow that a competitive equilibrium price vector \( p = (1, p^2) \) will exist for every positive \( p^m \).

The economy we have specified exhibits the following two equilibria:

\[
\begin{align*}
\text{Equilibrium} & & \text{Equilibrium 2} \\
p^m = 0, \ p = (1, 1) & & p^m = 1, \ p = (1, 4) \\
,x_1 = (x^1_1, x^1_2) = (8, 4) & & x_1 = (x^1_1, x^1_2) = (4, 3) \\
,x_2 = (x^2_1, x^2_2) = (16, 8) & & x_2 = (x^2_1, x^2_2) = (4, 8) \\
,x_3 = (x^3_1, x^3_2) = (4, 8) & & x_3 = (x^3_1, x^3_2) = (20, 9).
\end{align*}
\]

Claim 3. For this economy which satisfies the regularity assumptions of [1], there is no competitive equilibrium with \( p^m = 0.5 \).

Proof of Claim 3. For \( p^2 < 0.75 \), Mr. 1 has a negative level of wealth, so he cannot afford anything in his consumption set. This is inconsistent with equilibrium.

For \( p^2 \in [0.75, 3] \), we have \( x^1_1 = 16 \) and \( x^1_2 \geq 17 \), so there is excess demand for commodity one. When \( p^2 \in (3, 4] \) occurs, \( x^1_1 \geq 4 \) and \( x^1_2 \geq 26 \), so there is still excess demand for commodity one. Finally, for \( p^2 > 4 \), we have \( x^1_1 \geq 4 \) and \( x^1_2 > 20 \). Once again, there is excess demand for the numeraire commodity.

Since market clearing fails whenever \( p^2 \geq 0.75 \) holds and Mr. 1 goes bankrupt whenever \( p^2 < 0.75 \) holds, \( p^m = 0.5 \) is inconsistent with equilibrium.

However, by construction there is a competitive equilibrium with \( p^m = 1 \) and with \( p^m = 0 \). Hence we have

\[
0 \in \mathcal{E}^{\omega}(\omega, \tau),
\]

\[
0.5 \notin \mathcal{E}^{\omega}(\omega, \tau),
\]

and

\[
\in \mathcal{E}^{\omega}(\omega, \tau).
\]

This example thus establishes the existence of a “nice” economy in which \( \mathcal{E}^{\omega}(\omega, \tau) \) is not a connected set.
Concluding Remarks. Example 3 provides a nonpathological economy in which the set of equilibrium money prices is not an interval. We used preferences which generate vertical or horizontal segments of offer curves, merely as a simple way of guaranteeing that excess demand exists for certain prices. Any other preferences yielding excess demand for the appropriate range of prices would work just as well. However, Example 3 could be considered somewhat "unusual" in that it depends on powerful income effects. Increasing $p^m$ causes a redistribution of commodity one from consumer 1 to consumer 3. Eventually, consumer 1 cannot meet his tax obligations and there is no equilibrium. However, as $p^m$ is increased further, income effects cause relative prices to change, increasing the value of consumer 1's endowment by more than the additional tax. Equilibrium is restored when consumer 1's wealth becomes positive once again. This broken interval property is not limited to static models; see [4] for examples of infinite horizon overlapping generations economies (with and without balanced tax-transfer policies) for which the set of equilibrium money prices is not an interval.

References


Overlapping-generations models with money typically exhibit a multiplicity of equilibria in which the equilibrium price of money takes on a range of values. Balasko and Shell [3] find that for finite-horizon economies with a balanced tax-transfer policy, the set of equilibrium money prices contains an interval $[0, \bar{p}^m]$, where $\bar{p}^m$ is a positive scalar. If the tax-transfer policy is not balanced, the price of money must be zero. For infinite-horizon economies with a strongly balanced tax-transfer policy, the set of equilibrium money prices is not an interval.

* Support from the National Science Foundation under Grant SES-83-08450 is acknowledged. I would also like to thank Karl Shell for his advice and encouragement. All remaining errors are my own.
1 The tax-transfer policy $\tau = (\tau_0, \tau_1, \ldots, \tau_t)$ is said to be balanced if we have $\sum_{t=0}^T \tau_t = 0$. That is, a tax-transfer policy is balanced if the out-standing government debt is zero on the terminal date. See [3, Definition 3.1].
2 This follows immediately from Walras' law (see [3, Proposition 3.2]). The intuition is that all consumers are on their budget lines, so the value of aggregate excess demand equals the value of the money supply. In equilibrium, excess demand equals zero, so the value of the money supply is zero. Either the money supply is zero, and the tax-transfer policy is balanced, or the price of money is zero.
3 The tax-transfer policy $\tau = (\tau_0, \tau_1, \ldots, \tau_t, \ldots) \in R^T$ is said to be strongly balanced if there is an integer $t'$ with the property $\sum_{t=0}^T \tau_t = 0$ for $t \neq t'$.

See Balasko and Shell [2] for an analysis of the overlapping generations model with nominal taxes and transfers.

Non-connectedness of the Set of Equilibrium Money Prices: The Overlapping-Generations Economy

James Peck*

J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201

Received December 12, 1984; revised August 31, 1986

Consider the overlapping-generations economy with nominal taxes and transfers. Under some conditions, the set of equilibrium money prices is a non-negative interval. It has not been known whether this set can consist of two or more disjoint intervals. Three examples are provided here in which the set of equilibrium money prices is a non-connected set. The examples are for a finite-horizon balanced economy, an infinite-horizon balanced economy, and an infinite-horizon non-balanced economy. Journal of Economic Literature Classification Numbers: 021, 023.