

Comparing Sunspot Equilibrium and Lottery Equilibrium Allocations: The Finite Case^{*}

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July 22, 2003

Abstract

Sunspot equilibrium and lottery equilibrium are two stochastic solution concepts for nonstochastic economies. Recent work on nonconvex exchange economies has shown that when the randomizing device is a continuous random variable, applying the two concepts to the same fundamental economy yields the same set of equilibrium allocations. In the present paper, we examine economies based on a discrete randomizing device, which corresponds to the case where there is only a finite number of sunspot states. We extend the lottery model so that it can constrain the randomization possibilities available to agents in the same way that the sunspots model can. Every equilibrium allocation of our constrained lottery model has a corresponding sunspot equilibrium allocation. For almost all finite randomizing devices, the converse is also true. There are exceptions, however: for some randomizing devices, there exist sunspot equilibrium allocations with no lottery equilibrium counterpart.

* We thank Bob Anderson, Jess Benhabib, Alberto Bisin, Luca Bossi, David Easley, Richard Rogerson, Manuel Santos, Neil Wallace, Randy Wright, two anonymous referees, and especially Jim Peck for helpful comments and discussions. We also thank seminar participants at Arizona State, Cornell, the Federal Reserve Bank of Cleveland, ITAM, Ohio State, Stanford, Universitat Autònoma de Barcelona, UCSD, Washington University, the NBER General Equilibrium Conference, the Extrinsic Uncertainty Workshop at NYU, the Cornell/Penn State Macroeconomics Workshop, and the Meetings of the Society for the Advancement of Economic Theory. Part of this work was completed while Keister was visiting the University of Texas at Austin, whose hospitality and support are gratefully acknowledged.

1 Introduction

In nonconvex environments, the best allocations are often stochastic even if the economic fundamentals are non-stochastic. When a good is indivisible, for example, even risk averse consumers can benefit from the ability to purchase a contract that delivers the good with some probability, instead of having to choose between buying it with certainty or not at all. In such a situation, it is natural to use a stochastic equilibrium concept, even when the fundamentals of the economy are nonstochastic. That is, it is natural to introduce uncertainty that is *extrinsic* (*i.e.*, does not affect endowments, technologies, or preferences) and to allow agents to trade in contracts whose payoffs depend on the outcome of this uncertainty. This is precisely the approach taken in two well-known general equilibrium concepts: sunspot equilibrium, as introduced in Shell [15] and Cass and Shell [3], and lottery equilibrium, as introduced in Prescott and Townsend [11, 12].¹ Since the two models bring different approaches to bear on the same problem, it is natural to ask how their predictions compare: What is the relationship between the set of sunspot equilibrium allocations and the set of lottery equilibrium allocations for the same fundamental economy? We address this question for exchange economies where the number of consumers, the number of commodities, and the randomization possibilities are all finite.

Our focus in this paper is on economies with consumption nonconvexities: some goods are indivisible and consumers may be risk-loving. This represents a minimal departure from the standard Walrasian setting, yet allows stochastic trade to be useful. We assume that there are complete markets, with no restrictions to participation on those markets, that information is symmetric, and that the economic fundamentals are nonstochastic.² The fundamental economy is comprised of the set of consumers, together with their endowments and preferences, a (common) consumption set, and an extrinsic randomizing device. Both equilibrium concepts can be applied to the same fundamental economy, but they differ in the way trade is organized. In the sunspots model, extrinsic uncertainty is represented by a set of states of nature, and agents trade in state-contingent claims, such as “1 automobile to be delivered if state number 3 occurs.” Agents construct stochastic consumptions by purchasing different bundles to be delivered in different extrinsic states. In contrast, agents in the lottery model trade directly in probabilities, using assets such as “1 automobile to be deliv-

¹ See Prescott and Shell [10] for a discussion of the different histories of the two concepts and a survey of the relevant literature.

² Hence, if the fundamental economy were convex, the equilibria of the sunspots model would necessarily be non-stochastic. See the immunity theorem in Cass and Shell [3, Proposition 3].

ered with probability π .” In this way, agents directly purchase a probability distribution over their consumption set. No reference to a “state of nature” is made.

Shell and Wright [16] were the first to investigate the relationship between the two equilibrium concepts. They show how the equilibrium employment lotteries of Rogerson [14] can be implemented as sunspot equilibria, indicating that there is indeed a close connection. In addition, they show how sunspots can provide the necessary coordination to allocate indivisible goods among a finite number of consumers.³ Garratt [4] shows how the lottery model can be extended to economies with a finite number of consumers by including the coordination of individual lotteries in the market-clearing process. He then compares the equilibrium allocations of the lottery model with those generated by any sunspot variable with a finite number of states. He finds that every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation, but some sunspot equilibrium allocations have no lottery equilibrium counterpart. In other words, given a lottery equilibrium allocation, one can choose the sunspot variable in such a way that this same allocation is also part of an equilibrium of the sunspots economy. However, given a particular sunspot variable, there may be allocations that can be supported as a sunspot equilibrium but not as a lottery equilibrium. The critical difference here is that the sunspots model constrains consumers to choose a stochastic allocation that can be generated by the given sunspot variable, while in the lottery model consumers are free to choose any (affordable) stochastic allocation. This ability to constrain choice sets that is inherent in the sunspots model can lead to the existence of an equilibrium that is not present when the choice is unconstrained.⁴ A recent paper by Garratt, Keister, Qin, and Shell [6] (hereafter GKQS) examines the case where sunspot activity is represented by a continuous random variable, so that consumers have access to the same randomization possibilities in the sunspots model as in the lottery model. They show that in this case, the two models generate the same set of equilibrium allocations. The result is proven in a standard general equilibrium model with a finite number of consumers and a (possibly) nonconvex consumption set. Kehoe, Levine, and Prescott [9] show that this same result holds in a moral hazard economy with a continuum of consumers. In addition to its theoretical importance, the equivalence result has practical implications, as problems that are difficult to solve in one model may be more easily addressed

³ Prescott and Townsend [11, 12], Rogerson [14] and others study lottery economies with a continuum of consumers, where a law of large numbers implies that no coordination of individual lotteries is necessary.

⁴ This fact is also evident in Goenka and Shell [7], which introduces the concept of *robustness* of sunspot equilibria to refinements in the randomizing device. They show that, in nonconvex economies, not all sunspot equilibria are robust to refinements; some can be destroyed by giving consumers additional randomization possibilities.

in the other. For example, Garratt and Keister [5] show how an outstanding question regarding when sunspot equilibria are robust to refinements in the randomizing device is more easily solved by looking at the lottery formulation of the problem. In addition, when the consumption set has a finite number of elements, finding lottery equilibria reduces to solving a collection of linear programming problems, which can be computationally easier than solving the (nonlinear) sunspots model.

In this paper, we take a different approach. As in most of the sunspot equilibrium literature, we focus on the case where there is a finite number of sunspot states. There are good reasons for studying situations where the randomization possibilities available to consumers are constrained in some way. For example, transactions costs may prevent consumers from trading in a continuum of markets. Even in the absence of such costs, the government may place legal restrictions on the types of trading allowed (such as regulating risk classes in insurance) in order to achieve a preferred outcome. Because we work in environments where the first welfare theorem holds, constraining the randomization possibilities available to consumers cannot lead to a Pareto improvement. However, it typically does lead to a redistribution of resources, benefitting some consumers at the expense of others. We provide examples in Section 5 where using a constrained sunspot variable leads to the same equilibrium outcome as would using a continuous sunspot variable from a different set of endowments. In these examples, then, one can think of regulations on stochastic trade as a substitute for lump-sum taxes and transfers.⁵

Rather than adjusting the randomization possibilities in the sunspots model to match those in the lottery model, we modify the lottery model so that it can constrain consumer choice in the same way that the sunspot model does. We introduce the concept of *constrained lotteries*, under which only certain types of aggregate lotteries are possible and therefore only certain individual lotteries are available to consumers. We work in a completely finite environment – both the number of consumers and the number of possible lotteries is finite. We introduce lottery-producing firms that both generate individual lotteries and coordinate them to ensure feasibility. We present this extended model in Section 2 below. In Section 3, we show that the feasible allocations in our (constrained) lottery model are the same as those in the corresponding finite-state sunspots model. This is our first major point: The lottery model can be extended to completely finite economies.

⁵ While we consider the question of how an economy settles on a particular randomizing device to be extremely interesting, it is outside the scope of the present paper. We take the device as given and derive the implications of the two different models of stochastic trade based on this device.

Our goal is to compare the sets of equilibrium allocations generated by the two models in the finite environment. Because we have re-defined the lottery model, the results of Garratt [4] no longer apply. In particular, the sunspot equilibrium allocation that Garratt [4] shows to have no lottery equilibrium counterpart *does* have a counterpart in our constrained lottery model. In this sense, equating the randomization possibilities in the two models has again brought the sets of equilibrium allocations closer together. However, these sets are still potentially different because of a fundamental difference in the pricing systems. The two models define different objects to be the “basic” commodity of trade to which the law of one price applies. The lottery model directly assigns prices to probability distributions over the consumption set. This means that purchasing, say, a particular consumption bundle with probability one-half (and nothing otherwise) has a posted price. In the sunspots model, prices are assigned to states of nature. Two different states (or sets of states) with the same probability can have different contingent-commodity price vectors. Hence there need not be a unique cost for the lottery that delivers a particular consumption bundle with probability one-half (and nothing otherwise). In this way, some relative valuations of commodities in the sunspots model cannot be represented in the lottery model. However, the sunspots model also places certain restrictions on prices that are not present in the lottery model. To see this, suppose that there are three equally-likely states of nature and that the sunspots model assigns the same price vector to each of these states. Then the cost of receiving a particular bundle with probability two-thirds is twice the cost of receiving it with probability one-third, because the way a consumer constructs the two-thirds probability is by purchasing the same bundle in two states of nature. In the lottery model, however, there is no such restriction. The posted price of the bundle with two-thirds probability can be either more or less than twice the price of the same bundle with one-third probability. Thus, there are also relative valuations that can be represented in the lottery model but not in the sunspots model. These differences in price systems can potentially lead to differences in the sets of equilibrium allocations. If an equilibrium allocation in one model is supported only by a price system that cannot be represented in the other model, it seems likely that this allocation will not be an equilibrium in the other model.

This reasoning leads us to study in Section 4 the restrictions that equilibrium imposes on prices. For the sunspots model, we build on the results of GKQS [6] and show that any equilibrium allocation can be supported by a price system in which states with equal probability share the same (contingent-commodity) price vector. This eliminates some, but not all, of the additional flexibility

of prices in the sunspots model. To see where the remaining additional flexibility comes from, suppose that there are three states of nature, and that the first state has probability one-half while the second and third states each have probability one-quarter. Then our result says that any equilibrium allocation can be supported by prices in which the cost of receiving any bundle is the same in the second and third states. However, the result does *not* say that the cost of receiving the bundle in the first state is equal to the sum of the costs in the second and third states. In other words, in the sunspots model there can still be two different costs associated with receiving a particular bundle with probability one-half. Such a price system simply cannot be represented in the lottery model. For the lottery model, we show that the absence of arbitrage opportunities for lottery-producing firms requires that prices be linear in commodities and additive in the available randomization opportunities. This eliminates *all* of the additional flexibility of prices in the lottery model; any lottery price system that can arise in equilibrium necessarily has a representation in the sunspots model.

These pricing results combine to generate our central results comparing the sets of equilibrium allocations, which we present in Section 5. For every lottery equilibrium allocation, there is a corresponding sunspot equilibrium allocation. The converse of this statement is true *unless* the sunspot equilibrium allocation relies on the additional flexibility of sunspot prices, that is, unless the support prices cannot be translated into the lottery model. For almost all finite randomizing devices, this cannot happen and hence the two models lead to the same set of equilibrium allocations. However, for some randomizing devices the extra generality in the sunspot price system *does* matter and there can exist sunspot equilibrium allocations with no lottery equilibrium counterpart. We present two such examples. These examples show how the phenomenon described above, where consuming a given bundle with a given probability can have different costs depending on the set of states chosen, not only can arise in a sunspot equilibrium, but also can be critical for supporting a particular equilibrium allocation. Such an allocation cannot be supported as an equilibrium by the more restricted price system in the lottery model. The fact that the set of randomizing devices for which nonequivalence can occur has Lebesgue measure zero does not, of course, imply that such cases are unimportant in an economic sense. Hence the sets of sunspot equilibrium and lottery equilibrium allocations are often, but not always, equivalent in the model with a finite probability device.

2 The Two Models

We begin by describing the fundamental elements of the economy that are common to both models of trade. We then describe each model in detail.

2.1 The Environment

There is a finite set H of consumers, indexed (with a slight abuse of notation) by $h = 1, \dots, H$. There are L indivisible consumption goods, each of which can only be consumed in nonnegative integer amounts. There is a finite upper bound b_ℓ on the amount of good ℓ that may be consumed by any one consumer. These bounds allow us, for example, to study the case of $\{0, 1\}$ goods that has received so much attention in the literature on labor-market lotteries (i.e., Rogerson [14], Hansen [8], and Shell and Wright [16]). We also assume that there is a single divisible good (which we label good zero), so that the consumption set is given by

$$\mathcal{C} = C \times \mathbb{R}_+,$$

where C is a finite set with $K \equiv \prod_{\ell=1}^L b_\ell$ elements. We can think of the divisible good as “money” or “income spent on divisible goods.” The literature on lottery equilibrium often assumes that the consumption set has only a finite number of points, like our set C , because this simplifies notation, proofs, and computations (see Prescott and Townsend [11], Garratt [4], and Prescott and Townsend [13]). However, in the case of finite randomization possibilities this assumption would imply that consumers are locally satiated at every consumption bundle and would create equilibria where consumers do not spend all of their income. Such equilibria are of limited interest.⁶ By adding a divisible good, we eliminate these equilibria while maintaining much of the notational convenience of the finite-set approach. Adding more divisible goods would not change the analysis.

Each consumer has preferences represented by a Bernoulli utility function $U_h : \mathcal{C} \rightarrow \mathbb{R}$. To simplify the analysis in what follows, we assume that this function is additively separable in the divisible good, so that utility can be written as the sum of two functions u_h and v_h , with

$$u_h : C \rightarrow \mathbb{R} \quad \text{and} \quad v_h : \mathbb{R}_+ \rightarrow \mathbb{R}.$$

⁶ GKQS [6, Section 2.3] examine sunspots economies with finite randomization possibilities and provide results that apply only to equilibria satisfying certain conditions. The conditions deal with exactly this issue – they rule out equilibria that rely on satiation.

We assume that v_h is strictly concave for all h , so that consumers are risk averse in the divisible good. Consumer h has an endowment $e_h \in \mathbb{R}_+^L$ of the indivisible goods and an endowment $e_{0h} \in \mathbb{R}_+$ of the divisible good.

The set of *fundamentals* that is common to both models is the list $(\mathcal{C}, \{U_h, e_h, e_{0h}\}_{h \in H})$ plus a randomizing device, which represents the set of stochastic trades that agents can make. We find it helpful to think of this device as a roulette wheel. The wheel has on it a finite number M of slots, and the probability that the ball will fall into slot m is given by π_m . The wheel can be spun only one time, so a single spin represents all of the randomization possibilities available in the economy.⁷ As discussed above, there are practical reasons why the randomization possibilities available to consumers may be constrained. To keep things simple, however, we interpret the constraints as being technological in nature. The roulette wheel is, in our framework, the only way in which stochastic allocations can be generated.

To further simplify the notation, we do not allow stochastic allocations of the divisible good. This assumption is without any loss of generality, because the strict concavity of the functions v_h implies that an equilibrium allocation of either model cannot involve randomization in the assignment of the divisible good. The sole purpose of the divisible good is to provide consumers with a productive use for any “left over” income. Ruling out stochastic allocations of the divisible good actually complicates the specification of the sunspots model slightly, but it greatly simplifies the presentation of the lottery model.

Let F denote the set of feasible pure (nonstochastic) allocations of the endowments of the indivisible goods. Using $a = (a_h)_{h \in H}$ to denote a pure allocation with $a_h \in C$ for every h , we then have

$$F = \left\{ a \in C^H : \sum_{h \in H} a_h \leq \sum_{h \in H} e_h \right\}. \quad (1)$$

Both models generate equilibrium allocations that consist of a pure allocation of the divisible good paired with a probability distribution over the set F . The difference between them is the way in which stochastic trade is organized. We now describe the two models in detail.

⁷ If the wheel could be spun more than one time, we could always redefine the wheel so that each slot on the new wheel represents a sequence of realizations from the multiple spins of the old wheel. In this sense, allowing only a single spin is without loss of generality.

2.2 The Sunspots Model

In the sunspots model, each slot on the roulette wheel is marked with a number and called a “state of nature.” Consumers then trade in state-contingent claims on each indivisible good. Formally, the randomizing device is now represented by a probability space (S, Σ, π) . Here S is a finite set with M elements and Σ is the set of all subsets of S . The mapping between the roulette wheel and this probability space is straightforward; if slot m is labelled as state s , then the probability of this state is given by $\pi(s) \equiv \pi_m$. We also use $\pi(A)$ to denote the probability of any subset A of S .

Let X be the set of functions $x_h : S \rightarrow C$, that is, the set of allowable, stochastic individual consumption plans for the indivisible goods. Prices for the indivisible goods are given by a function $p : S \rightarrow \mathbb{R}_+^L$.⁸ We take the divisible good to be the numeraire, and we use x_{0h} to denote consumer h 's (certain) consumption of this good. Consumer h chooses her consumption plan to solve

$$\begin{aligned} & \max_{x_h, x_{0h}} \sum_{s \in S} \pi(s) u_h(x_h(s)) + v_h(x_{0h}), \\ & \text{subject to } \sum_{s \in S} p(s) \cdot x_h(s) + x_{0h} \leq \sum_{s \in S} p(s) \cdot e_h + e_{0h}, \\ & \quad x_h \in X, \quad x_{0h} \in \mathbb{R}_+. \end{aligned} \tag{2}$$

Let X^H be the set of functions $x : S \rightarrow \mathbb{R}^{LH}$ such that $x_h^* \in X$ for all h . The definition of equilibrium for the sunspots economy is as follows.

Definition 1 A *sunspot equilibrium* consists of a price function $p^* : S \rightarrow \mathbb{R}_+^L$ and an allocation $(x^*, x_0^*) \in X^H \times \mathbb{R}_+^H$ such that

(i) Given p^* , (x_h^*, x_{0h}^*) solves the consumer's problem (2) for each $h \in H$,

and

(ii) (x^*, x_0^*) is feasible, i.e., we have $x^*(s) \in F$ for all $s \in S$ and $\sum_{h \in H} x_{0h} \leq \sum_{h \in H} e_{0h}$.

Condition (ii) shows how an equilibrium allocation x^* of the indivisible goods generates a probability distribution over the set F defined in (1). The distribution is given by $\pi \circ (x^*)^{-1}$; in other words, to every allocation $a \in F$, it assigns the total probability of the set of states s such that $x^*(s) = a$. This distribution, together with the allocation of the divisible good, summarizes an equilibrium allocation of the sunspots model.

⁸ In contrast to the notation in GKQS [6], the prices p here are the actual contingent-commodity prices. They are *not* probability-adjusted prices.

This specification of the sunspots model is fairly standard. We now turn our attention to the lottery model, which we must modify to allow for constraints on the randomization possibilities.

2.3 The Lottery Model

Following Prescott and Townsend [11, 12], we treat each consumption bundle in C as a separate commodity. This means there are K commodities (plus the divisible good). A quantity of commodity k corresponds to the probability of receiving bundle c_k . In this way, consumers in the lottery model directly choose probability distributions over C , which are called individual lotteries. However, in the constrained lottery model consumers are not able to purchase an arbitrary probability distribution over C . Rather, they are only able to choose lotteries that can be generated by the pre-specified randomizing device, which we again interpret as a roulette wheel. Unlike in the sunspots model, in the lottery model consumers are not able to specify how their possible consumption bundles are arranged on the wheel. As an example, suppose the wheel has two equally-likely slots (black and red) and that a consumer purchases a lottery that delivers a particular bundle with probability one-half. The consumer is not able to specify whether she will receive that bundle when the black slot is realized or when the red slot is realized. If she could, she would be buying state-contingent commodities and we would be back in the sunspots model. Instead, the task of arranging the demanded lotteries on the wheel so that they meet feasibility requirements is left to the wheel's operator; we discuss this process in detail below.

2.3.1 Constrained Lotteries and Consumer Choice

An individual lottery is a probability distribution δ_h over C . Because C is a finite set, this distribution is a vector $(\delta_h(c_1), \dots, \delta_h(c_K)) \in \mathbb{R}_+^K$ whose elements sum to unity, where $\delta_h(c_k)$ is the probability assigned to the consumption bundle c_k . Let $\Delta(C)$ denote the set of all probability distributions over the bundles of indivisible goods; this is equivalent to the $(K - 1)$ -dimensional unit simplex in \mathbb{R}^K . Only a few of these lotteries can actually be generated by the roulette wheel. Consider a function g_h that assigns an element of C to each of the M slots on the wheel. Abusing notation slightly, let M also represent the set of slots on the wheel, so that we have $g_h : M \rightarrow C$. Let G be the set of all such functions. (Note that since M and C are finite sets, G is also a finite set.) Then the set of probability distributions over C that can be generated by a particular randomizing device is given by

$$\Gamma(C) = \left\{ \delta_h \in \Delta(C) : \exists g_h \in G \text{ such that } \left(\delta_h(c_k) = \sum_{m: g_h(m)=c_k} \pi_m \right) \text{ holds for all } k \right\}.$$

In other words, an individual lottery δ_h is in the set $\Gamma(\Delta)$ if there is some way of arranging consumption bundles on the roulette wheel so that the probability distribution over C generated by a spin of the wheel is exactly δ_h . As an example, suppose that $K = 3$ and that there are 3 equally-likely slots on the wheel. Then $\Delta(C)$ is the triangular simplex shown in Figure 1. The set $\Gamma(C)$ contains those lotteries in which the probability placed on each consumption bundle is a multiple of one-third; these lotteries are represented by the ten dots in the figure. Each consumer must choose one of these lotteries. In this way, the consumer is constrained to choose a lottery that is “available in the market.” It is important to recognize that this is exactly the set of probability distributions over C that a consumer in the sunspots model can construct using state-contingent commodities when there are three equally-likely states. This is what we mean when we say that our (constrained) lottery model places the same restrictions on consumer choice that the finite-state sunspots model does.

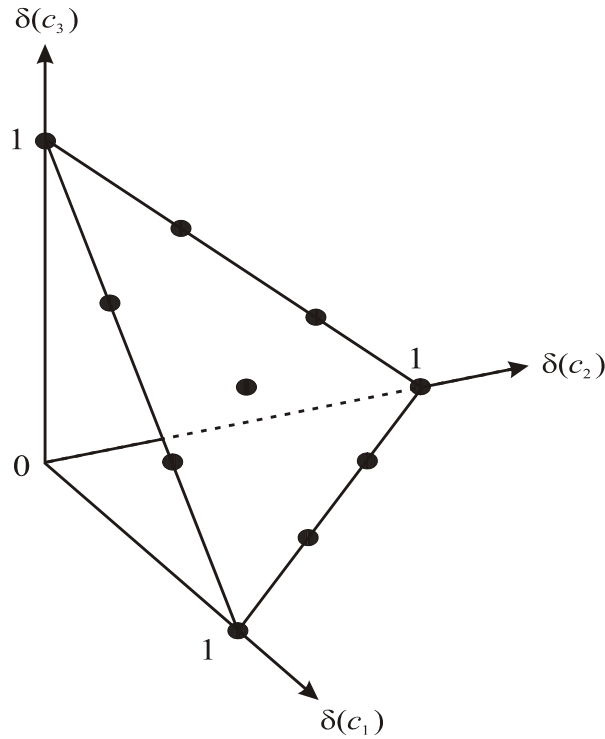


Figure 1: The set of possible individual lotteries

An individual consumption plan is a pair (δ_h, c_{0h}) specifying a lottery over C and a (certain) amount of the divisible good. We again take the divisible good to be the numeraire. A general formulation of lottery prices is then a function $\phi : \Gamma(C) \rightarrow \mathbb{R}_+$, that is, a function that assigns a price (in units of the divisible good) to each possible lottery over the indivisible goods. Using this formulation, we can write the consumer's lottery-choice problem as

$$\begin{aligned} & \max_{\delta_h, c_{0h}} \sum_{k=1}^K \delta_h(c_k) u_h(c_k) + v_h(c_{0h}), \\ & \text{subject to } \phi(\delta_h) + c_{0h} \leq \phi(e_h) + e_{0h}, \\ & \delta_h \in \Gamma(C), \quad c_{0h} \in \mathbb{R}_+, \end{aligned} \tag{3}$$

where e_h here represents the degenerate lottery that gives consumer h 's endowment of the indivisible goods with probability one.

2.3.2 Lottery-Producing Firms

Lotteries are produced by a representative, competitive firm that has access to the randomization technology represented by the roulette wheel. The firm operates by buying and selling “lottery tickets,” where each ticket entitles the holder to a particular lottery. Let j index the lotteries that the firm can produce, so that δ^j is a typical element of $\Gamma(C)$. Let $y(\delta^j)$ denote the firm's sales of tickets promising the distribution δ^j . This number may be positive or negative (with negative sales indicating purchases), but must be an integer. Thus, a production plan for the firm is a function $y : \Gamma(C) \rightarrow \mathbb{Z}$, where \mathbb{Z} is the set of integers. The firm must, of course, choose a feasible production plan. That is, it must be able to arrange the lotteries that it buys and sells on the roulette wheel in such a way that, for all realizations, it gives away no more resources than it takes in. Let n index the individual tickets of lottery j bought or sold by the firm, so that we have $n = 1, \dots, |y(\delta^j)|$. Each ticket is therefore identified by a pair (j, n) , indicating the type of lottery it delivers (j) and the “serial number” of the ticket within that type (n). We must keep track of identical lottery tickets individually here because they may need to be generated by different slots on the roulette wheel. A function $g_{j,n}$ that assigns the distribution promised by lottery ticket (j, n)

to spaces on the lottery wheel has the form $g_{j,n} : M \rightarrow C$,⁹ with

$$\delta^j(c_k) = \sum_{m: g_{j,n}(m)=c_k} \pi_m \quad \text{for all } k. \quad (4)$$

For all types of lotteries j , define

$$I_j = \begin{cases} -1 & \text{if } y(\delta^j) < 0 \\ 0 & \text{if } y(\delta^j) = 0 \\ 1 & \text{if } y(\delta^j) > 0 \end{cases}$$

Then I_j is an indicator for whether the firm is buying or selling lotteries of type j . Feasibility requires that the functions $g_{j,n}$ used to create the individual lottery tickets satisfy

$$\sum_{j,n} I_j g_{j,n}(m) \leq \mathbf{0} \quad \text{for all } m. \quad (5)$$

Note that this is a vector inequality: for each slot m , the net amount of each good that the firm must deliver when m is realized must be non-positive. Conditions (4) and (5) together are equivalent to saying that each of the individual lottery tickets must be a marginal distribution of some common joint lottery over the set of feasible (pure) allocations F .¹⁰ The production set of the firm is

$$Y = \left\{ y : \Gamma(C) \rightarrow \mathbb{Z} : \exists \{g_{j,n}\}_{j,n} \text{ such that (4) holds for each } (j,n) \text{ and (5) is satisfied.} \right\}.$$

The firm's problem is given by

$$\max_{y \in Y} \sum_{\delta \in \Gamma(C)} \phi(\delta) y(\delta). \quad (6)$$

Before proceeding, it may be helpful to look at a simple example that illustrates how our firms differ from those used in the previous literature (see especially Rogerson [14]).

Example: $C = \{0, 1\}$. There is a single indivisible good that can only be consumed in either one unit or not at all. Suppose that all consumers are identical and that prices are such that everyone

⁹ This notation is slightly different from our earlier use of subscripts to the function g . Instead of indexing by the purchaser of the lottery ticket, we are now indexing by type of lottery and serial number of the ticket. Of course, this is not important; it only serves to simplify the notation. Regardless of how we index the lotteries, g will always represent a function that assigns consumption bundles to slots on the wheel.

¹⁰ The idea of using aggregate or joint lotteries to ensure coordination when there is a finite number of consumers was introduced in Garratt [4], where coordination is provided by the Walrasian auctioneer as a part of the market-clearing process.

demands the lottery that gives one unit of the good with probability two-thirds and nothing with probability one-third. First consider the case where there is a continuum of consumers and randomization is unconstrained. Then the firm buys two-thirds of a unit of the good per consumer, and sells the demanded lottery. Each consumer comes to the firm and a weighted coin is tossed to see if the consumer receives the good or not. Since the coin tosses are independent across consumers,¹¹ two-thirds of the consumers will receive the good and hence this plan is feasible. This is the “traditional” approach to lottery-producing firms.

Next, let’s look at how the firm in our model operates. Suppose that there are three consumers and the roulette wheel has three equally-likely slots. The firm faces the above demand conditions – all consumers want to receive the good with probability two-thirds. If the firm buys two units of the indivisible good (or, more precisely, two units of the degenerate lottery that delivers one unit of the good with probability one), it can offer three units of the demanded lottery. Consider the joint lottery

Roulette wheel slot:	<u>red</u>	<u>black</u>	<u>green</u>
ticket-holder 1:	1	1	0
Payoff to ticket-holder 2:	1	0	1
ticket-holder 3:	0	1	1

Looking at the columns shows that, regardless of which slot is realized, the firm pays out two units of the good (exactly equal to the resources it purchased). Looking at the rows shows that each ticket-holder receives the good with probability two-thirds, as desired.

Two comments are in order here. First, the plan of the finite-case firm only works if the number of tickets it sells is a multiple of three. There is no way it can sell four of these lotteries, for example, because there is no joint lottery that would generate four such marginal distributions. (See Shubik [17] on this phenomenon.) Outside of this restriction, the finite-case firm behaves very much like the continuous-case one: it purchases two-thirds of a unit of the good per customer and delivers the good with probability two-thirds to each customer. Second, suppose that the roulette wheel has only two slots, with probabilities one-third and two-thirds. Now the plan of the finite-case firm is infeasible, since there is no longer a joint lottery that gives the same marginal distribution to every ticket-holder. That such seemingly small changes in the randomizing device can have important effects is a recurrent theme in the finite case.

¹¹ This statement ignores problems associated with integrating over a continuum of i.i.d. random variables.

2.3.3 Equilibrium

The definition of equilibrium in our constrained lottery economy is the following.

Definition 2 A lottery equilibrium consists of a price function $\phi^* : \Gamma(C) \rightarrow \mathbb{R}_+$ and an allocation $(\delta^*, c_{0h}^*, y^*)$ such that

(i') Given ϕ^* , (δ_h^*, c_{0h}^*) solves the consumer's lottery problem (3) for each $h \in H$,

(ii') Given ϕ^* , y^* solves the firm's problem (6),

and

(iii') We have

$$y^*(\delta) = \sum_{h \in H} \left(I_{(\delta_h^* = \delta)} - I_{(e_h = \delta)} \right) \text{ for every } \delta \in \Gamma(C)$$

and

$$\sum_{h \in H} c_{0h}^* \leq \sum_{h \in H} e_{0h}.$$

Condition (iii') is the market-clearing constraint. It requires that the number of units of each lottery that the firm produces be equal to the net demand for the lottery by households (here I is the standard indicator function), and that the market for the divisible good clears.

Recall that a sunspot equilibrium allocation generates a probability distribution over the set F of (pure) feasible allocations of the indivisible goods. The same is true in the lottery model. Let g_h denote the arrangement function for the lottery δ_h^* . Define g to be the vector-valued function comprised of the functions g_h , so that we have $g : M \rightarrow C^H$. The feasibility condition (5) guarantees that for every realization of the wheel, the assignment of resources is feasible. Therefore we have $g(m) \in F$ for all m . Then $\pi \circ g^{-1}$ is the probability distribution over F (or, the joint lottery) associated with the lottery equilibrium allocation. This relates the market-clearing conditions here to those in Garratt [4] and GKQS [6]. In those papers, market-clearing is stated directly in terms of the joint lottery. This is because the joint lottery is arranged by the auctioneer (and hence is naturally considered part of the market-clearing process). Here the joint lottery is arranged by the firm and hence is determined by the firm's equilibrium production plan.

3 Comparing The Models

While the two models are stated in very different terms, the literature (beginning with Shell and Wright [16]) has shown they have much in common. Introducing constrained lotteries has, for the finite case, brought them closer still. In this section, we compare the sets of feasible allocations and of possible prices in the two models. We show that the sets of feasible allocations are identical. In this sense, our definition of constrained lotteries places the “right” restrictions on stochastic allocations in the lottery model. The price systems, on the other hand, are fundamentally different.

3.1 Corresponding Allocations

Because an individual consumption plan is a function in the sunspots model and a probability distribution in the lottery model, we need to be precise about how we compare these objects. At the individual level, a sunspot consumption plan (x_h, x_{0h}) induces a lottery consumption plan (δ_h, c_{0h}) through the equations

$$\delta_h(c_k) = \pi \circ x_h^{-1}(c_k) \quad \text{for all } k, \text{ and} \quad (7)$$

$$c_{0h} = x_{0h}. \quad (8)$$

In other words, the probability assigned by the individual lottery δ_h to the bundle of indivisible goods c_k is equal to the probability assigned by π to the set of states in which x_h delivers c_k . Note that $x_h \in X$ holds if and only if $\delta_h \in \Gamma(C)$ holds, that is, the lottery δ_h defined in (7) is individually feasible in the lottery model if and only if the plan x_h generating it is individually feasible in the sunspots model. At the aggregate level, a sunspot allocation (x, x_0) induces a lottery allocation (δ, c_0, y) in two steps. First, the individual sunspot consumption plans induce the individual lotteries through (7). The production plan y is then given by

$$y(\delta) = \sum_{h \in H} (I_{(\delta_h = \delta)} - I_{(e_h = \delta)}) \quad (9)$$

for all $\delta \in \Gamma(C)$. In other words, y is the unique production plan that makes the consumption allocation feasible. We now show that, through this relationship, feasible sunspot allocations always correspond to feasible lottery allocations and *vice versa*.

Proposition 1 *A sunspot allocation (x, x_0) is feasible if and only if the corresponding lottery allocation (δ, c_0, y) given by (7), (8), and (9) is feasible.*

The proofs of all propositions are contained in the appendix. The intuition for this result is fairly straightforward. If (x, x_0) is a feasible sunspots allocation, then by definition it is possible to arrange the individual consumption plans x_h on the wheel in such a way that a feasible (pure) allocation is assigned to every slot. This same arrangement pattern can then be used in the lottery model to construct assignment functions $g_{j,n}$ that show how the production plan y defined in (9) is contained in the production set Y . Conversely, if (δ, c_0, y) is a feasible lottery allocation, one can use the functions $g_{j,n}$ to show that the sunspot consumption plans x_h are such that $x(s) \in F$ holds for every s .¹² It should come as no surprise that the sets of feasible allocations are the same in the two models. Our goal in defining the concept of constrained lotteries was precisely to allow only those allocations that are feasible in the finite-state sunspots model.

3.2 Corresponding Prices

Although both models generate the same set of feasible allocations, there is an important difference in the way they assign prices. Essentially, the two models define different objects to be the basic commodity of trade to which the law of one price applies. This can be seen in the context of a simple example. Suppose the roulette wheel has two equally-likely slots. Pick an arbitrary consumption bundle c_k and ask: What is the price of receiving that bundle with probability one-half and nothing otherwise? In the lottery model, there is a unique answer: $\phi(\delta)$, where δ represents the specified lottery. In the sunspots model, however, the answer is either $[p(1) \cdot c_k]$ or $[p(2) \cdot c_k]$, depending on the state in which the bundle is purchased. Hence the sunspots model has the ability to assign different prices to something that the lottery model considers a single commodity. Now ask: What is the relationship between the cost of buying a bundle with probability one-half (and nothing otherwise) and the cost of buying it with probability one? In the sunspots model, there is a unique answer: the cost of buying c_k with probability one is given by $[p(1) + p(2)] \cdot c_k$. In the lottery model, however, no such relationship need hold between $\phi(\delta)$ and $\phi(\delta')$, where δ' represents the (degenerate) lottery giving c_k with probability one. In this way, the lottery model has more flexibility in the assignment of prices across different probabilities.

If we have a sunspot equilibrium allocation, we know that the corresponding lottery allocation

¹² There is an additional complication in this direction: different sunspot allocations may correspond to the same lottery allocation. Therefore matching the functions g_h with the functions x_h may first require “relabelling” the states of nature in such a way that the probability of receiving each consumption bundle is preserved. See GKQS [6] for an extensive discussion of this many-to-one relationship.

is feasible. However, if the support prices in the sunspots model cannot be translated into a price function in the lottery model, it seems likely that the allocation will *not* be part of an equilibrium of the lottery model. (We show that this is indeed the case in Section 5.) Therefore, the critical issue is under what conditions the prices in one model can be translated into corresponding prices in the other model. Some additional notation is needed. Let Γ denote the image of $\Gamma(C)$ in the interval $[0, 1]$. Then Γ is the set of probabilities that can be assigned to consumption bundles in a lottery generated by the roulette wheel. For example, if there are three equally-likely slots, then we have $\Gamma = \{0, 1/3, 2/3, 1\}$. The same is true if there are two slots with probabilities one-third and two-thirds; different randomizing devices can lead to the same set Γ . Note that we have the relationship

$$\Gamma = \left\{ \theta \in [0, 1] : \theta = \sum_{s \in A} \pi(s) \text{ for some } A \subseteq S \right\}.$$

In other words, a consumption bundle in the lottery model can be purchased with probability θ if and only if it can be purchased in the sunspots model in a set of states whose total probability is θ .

We now introduce a condition that is central to determining the relationship between the equilibria of the two models. Suppose a sunspot price function p has the following property: for any two disjoint subsets A and B of S , we have

$$\sum_{s \in A} \pi(s) = \sum_{s \in B} \pi(s) \quad \text{implies} \quad \sum_{s \in A} p(s) = \sum_{s \in B} p(s). \quad (10)$$

In other words, suppose that whenever there are different ways of combining states together to get the same probability, the total cost of purchasing any consumption bundle in either of the two sets of states is the same. When this is true, the sunspots model assigns a unique price vector to each level of probability in Γ , and therefore the sunspot prices can be transformed into lottery prices. Let \widehat{P} be the set of functions $p : S \rightarrow \mathbb{R}_+^L$ satisfying condition (10).

Definition 4 Given a sunspot price function $p \in \widehat{P}$, the *corresponding lottery price function* $\phi : \Gamma(C) \rightarrow \mathbb{R}_+^L$ is defined in two steps. First, for all $\theta \in \Gamma$, define $q : \Gamma \rightarrow \mathbb{R}_+^L$ by

$$q(\theta) = \sum_{s \in A} p(s) \quad (11)$$

for any set $A \subseteq S$ satisfying $\sum_{s \in A} \pi(s) = \theta$. Then the price function ϕ is given by

$$\phi(\delta) = \sum_{k=1}^K q(\delta(c_k)) \cdot c_k. \quad (12)$$

The function q assigns a price vector to each probability in Γ , with $q(0)$ naturally equal to the zero vector. Any consumption bundles that are purchased with probability θ are then priced according to $q(\theta)$. Only when sunspot prices satisfy (10) will the function q , and hence the lottery prices, be well defined. If the sunspot price function does not satisfy (10), we say that it has *no corresponding lottery prices*. For a given lottery price function ϕ , there may or may not exist functions p and q that generate ϕ through (11) and (12). For example, lottery prices must be linear in commodities for (12) to hold. In addition, certain restrictions regarding the assignment of prices across different probabilities must be satisfied. If, for a given lottery price function ϕ , there does not exist a pair (p, q) satisfying (11) and (12), we say that ϕ has *no corresponding sunspot prices*.

To see what restrictions must be placed on prices in order for them to translate from one model to the other, consider the case of three equally-likely slots on the wheel. In order for a sunspot price function p to have corresponding lottery prices in this case, it must assign a unique price to receiving any consumption bundle with probability two-thirds. This implies that we must have

$$p(1) + p(2) = p(1) + p(3) = p(2) + p(3),$$

which in turn implies that $p(1) = p(2) = p(3)$ must hold. In this case, sunspot prices would need to be constant across states in order to have corresponding lottery prices. Conversely, any lottery price function ϕ must be linear in quantities and have a price vector q that satisfies

$$q\left(\frac{2}{3}\right) = 2q\left(\frac{1}{3}\right) \quad \text{and} \quad q(1) = 3q\left(\frac{1}{3}\right)$$

in order to correspond to some sunspot prices. Now suppose that we look at another randomizing device with only two slots on the wheel, with probabilities one-third and two-thirds. In this case there is no restriction on the sunspot price function p . The only restriction on the lottery price function is

$$q(1) = q\left(\frac{1}{3}\right) + q\left(\frac{2}{3}\right).$$

Hence the strength of the restrictions required for prices to translate from one model to the other depend on the details of the randomizing device. Many restrictions are needed when slots are equally likely, while fewer are required when each slot has a distinct probability.

4 Equilibrium Prices

In this section we show that, in both models, equilibrium places important restrictions on the form on the price function. These restrictions have the effect of bringing the sets of possible price functions closer together. For the sunspots model, we show that every equilibrium allocation can be supported by prices in which equal-probability states share the same contingent-claims price vector. This removes some, but not all, of the extra pricing flexibility in the sunspots model. In the lottery model, we show that if a price function does not generate arbitrage opportunities for the lottery-producing firm, then it has a corresponding sunspots price function. Hence, equilibrium considerations eliminate all of the extra pricing flexibility in the lottery model.

4.1 Sunspot Equilibrium Prices

We first prove a result for the sunspots model. GKQS [6] show that with a finite number of equally-likely states, any equilibrium allocation can be supported by prices that are constant across states. We extend this result to the case where only some states are equally likely. We show that any equilibrium allocation can be supported by prices in which $\pi(s) = \pi(s')$ implies $p(s) = p(s')$ for any states s and s' . In other words, equal probability states share the same vector of contingent-claims prices. As shown in the previous section, such a result is critical for supporting the corresponding allocation as an equilibrium of the lottery model, since the lottery model necessarily assigns a single price to receiving a bundle with probability $\theta = \pi(s)$.

We begin by defining sets of individual consumption plans that are in some sense equivalent. Suppose that we have a subset $A \subseteq S$ of states, each of which has the same probability. Let N_A be the number of states in A , and suppose (without any loss of generality) that these states are consecutively numbered, beginning with state 1.

Definition 5 For any $x_h \in X$ and any set A of equi-probable states, the *A-shift class* of x_h , denoted

$T_A(x_h)$, is the set of plans x_h^t such that

$$x_h^t = \left\{ \begin{array}{ll} x_h(s+t) & \text{for } s \in A \\ x_h(s) & \text{for } s \notin A \end{array} \right\}$$

holds for some $t \in \{0, 1, \dots, N_A\}$, where the addition is modulo N_A .

The idea here is simply to “shift” the consumption bundles across the equally-probable states of nature. If these states were not consecutively numbered, the idea would be exactly the same, only the notation would be more complicated.

The A -shift class is a set of consumption plans among which a consumer with von Neumann-Morgenstern preferences would clearly be indifferent.¹³ We next show that this fact has important implications for the form of equilibrium prices. For any price vector p and any equi-probable set A , define another price vector \bar{p}_A by

$$\bar{p}_A = \left\{ \begin{array}{ll} \frac{1}{N_A} \sum_{s \in A} p(s) & \text{for } s \in A \\ p(s) & \text{for } s \notin A \end{array} \right\}.$$

The price vector \bar{p}_A replaces prices of states in A with the average price across those states. We show that if p^* supports some allocation (x^*, x_0^*) as a sunspot equilibrium, then that same allocation is also supported as an equilibrium by \bar{p}_A^* .

Proposition 2 *Suppose (p^*, x^*, x_0^*) is a sunspot equilibrium. Then for any set of equi-probable states A , $(\bar{p}_A^*, x^*, x_0^*)$ is also a sunspot equilibrium.*

The proof of this proposition involves showing that when prices p^* are replaced with \bar{p}_A^* , the optimal choice of each consumer stays the same. One way of interpreting the result is as saying that, in equilibrium, no state can trade at a meaningful discount relative to another state with the same probability. Two equally-likely states can have different contingent-claims price vectors, but it must be the case that if we average out these differences by switching to prices \bar{p}_A^* , no consumer wishes to change her consumption plan.

This result can be used to give sufficient conditions on the randomizing device to guarantee that sunspot prices have corresponding lottery prices. As an example, suppose that there are three states of nature, with $\pi(1) = \pi(2) = 1/6$ and $\pi(3) = 2/3$. Then Proposition 2 says that any equilibrium allocation can be supported by a price function that assigns a unique cost to receiving a particular bundle with probability one-sixth. The other possible probabilities (one-third, two-thirds, five-

¹³ Our results would actually apply to a broader class of preferences, including those defined by Balasko [1].

sixths, and one) then have unique costs as well. Therefore condition (10) is satisfied, and the corresponding lottery prices are given by (11) and (12). In addition, when there are multiple sets of equi-probable states, the proposition applies to each of them. Suppose there are four states with probabilities $\pi(1) = \pi(2) = 1/10$ and $\pi(3) = \pi(4) = 2/5$. Then Proposition 2 says that any equilibrium allocation has support prices with $p(1) = p(2)$ and $p(3) = p(4)$. From this it follows that each of the other possible probabilities in the set Γ has a unique cost, and therefore (10) is again satisfied.

Notice, however, that the conclusion in Proposition 2 applies only to equally-likely states, not to sets of states that add to the same total probability. Suppose, for example, that there are three states with probabilities $\pi(1) = 1/2$ and $\pi(2) = \pi(3) = 1/4$. Then the proposition tells us that any equilibrium allocation has support prices with $p(2)$ equal to $p(3)$, but it does not say anything about the relationship between $p(1)$ and the sum of $p(2)$ and $p(3)$. This distinction will be critical in our examples in the next section of sunspot equilibrium allocations with no lottery equilibrium counterpart.

4.2 No-Arbitrage Lottery Prices

In the lottery model, arbitrage arguments based on the constant-returns-to-scale nature of the firm's production technology can be used to place restrictions on the set of prices that could appear in equilibrium. Garratt [4] demonstrates that in a model where the auctioneer coordinates individual lotteries, the absence of arbitrage requires that lottery prices be linear in the underlying goods. This is also true under our specification of the lottery-producing firm.

Proposition 3 *If there exists a solution to the firm's problem (6), then there exists a function $q : \Gamma \rightarrow \mathbb{R}_+^L$ such that (12) holds for every $\delta \in \Gamma(C)$.*

The intuition for this result is conveyed in a simple example. Suppose there are two indivisible goods, and let θ denote the probability associated with any one slot on the wheel. One possible lottery delivers two units of each good with probability θ (and nothing otherwise). The firm can produce this lottery using two units each of the lottery that delivers one unit of the first good with probability θ and the lottery that delivers one unit of the second good with probability θ . The firm can also reverse this production plan, using the single lottery as an input and the four smaller lotteries as outputs. Hence the price of the former must exactly equal the sum of the prices of the latter in order for the firm not to have an arbitrage opportunity. This implies that for each $\theta \in \Gamma$,

there exists a vector $q(\theta)$ such that the price of receiving bundle c_k with probability θ is given by the inner product $q(\theta) \cdot c_k$. A similar argument can be used to show that the cost of a lottery that delivers c_k with probability θ and $c_{k'}$ with probability θ' must be equal to the sum of $q(\theta) \cdot c_k$ and $q(\theta') \cdot c_{k'}$.

The previous literature has assumed that lottery prices are also linear in probabilities, that is, the price of probability θ on any bundle c_k is given by $\theta\psi \cdot c_k$ for some $\psi \in \mathbb{R}_+^\ell$. This is a much stronger restriction on prices than that given in Proposition 3. Neither prices in the sunspots model nor those in the lottery model need be linear in probabilities. However, as we demonstrate next, lottery prices must satisfy some restrictions across probabilities in order to prevent the firm from having arbitrage opportunities.

Definition 6 For a given randomizing device, a lottery price function $q : \Gamma \rightarrow \mathbb{R}_+^L$ is *additive* if for any set $A \subseteq M$ of slots on the wheel, we have

$$q\left(\sum_{m \in A} \pi_m\right) = \sum_{m \in A} q(\pi_m).$$

This definition does *not* imply that prices are linear in probabilities. Suppose there is a single commodity and that there are two slots on the wheel, with probabilities one-third and two-thirds. Then the following price function is additive:

$$q\left(\frac{1}{3}\right) = \frac{1}{4}, \quad q\left(\frac{2}{3}\right) = \frac{3}{4}, \quad q(1) = 1.$$

In this example, $q(1)$ gives the price of receiving one unit of the good with certainty, which can be generated by adding together the probabilities of the two individual slots. Additivity therefore requires that we have

$$q(1) = q\left(\frac{1}{3}\right) + q\left(\frac{2}{3}\right),$$

but does not impose any particular relationship between $q(1/3)$ and $q(2/3)$. Our next result is that no-arbitrage prices must be additive.

Proposition 4 *If there exists a solution to the firm's problem (6), then lottery prices are additive.*

The intuition for this result is very similar to that for Proposition 3. Let θ and θ' denote the

probabilities of any two slots on the wheel. One possible production plan is the following: buy some bundle c_k with probability θ , also buy that same bundle with probability θ' , and sell c_k with probability $(\theta + \theta')$. If lottery prices are not additive this plan will generate nonzero profits, and hence the firm could generate unbounded profits by replicating either this plan or the negative of it.

For the case where all slots on the wheel are equally likely, every probability in Γ is a multiple of $1/M$. Therefore, for any $\theta \in \Gamma$, additivity implies that we have

$$q(\theta) = q\left(\frac{\alpha}{M}\right) = \alpha q\left(\frac{1}{M}\right)$$

for some integer α . In other words, in this case prices must be linear in probabilities. We state this as a corollary.

Corollary 1 *Suppose the randomizing device has M equally-likely slots. Then if the firm's problem (6) has a solution, lottery prices must be linear in probabilities, i.e., the price of individual lottery δ_h must be given by*

$$q(\delta_h) = q(1) \cdot \sum_{k=1}^K \delta_h(c_k) c_k.$$

The most important result in this subsection is that *every* no-arbitrage lottery price function has a corresponding sunspot price function. While in principle there are many lottery prices that cannot be represented in the sunspots model, none of them can ever support an equilibrium allocation in the lottery model. To see why this is true, recall that a lottery price function has a corresponding sunspot price function if there exist functions p and q such that (11) and (12) hold. The q function satisfying (12) is given by Proposition 3, which tells us that no-arbitrage lottery prices must be linear in commodities. The function p is constructed in the following way. For every state s in the sunspots model, set $p(s) = q(\theta)$, where $\theta = \pi(s)$. This rule completely defines the sunspot price function p . What remains is to show that p does in fact generate q through (11), that is, that for every $\theta \in \Gamma$, we have

$$q(\theta) = \sum_{s \in A} p(s)$$

for every set $A \subseteq S$ satisfying $\sum_{s \in A} \pi(s) = \theta$. This relationship is ensured by the fact that q is additive with respect to the given randomizing device. Each $p(s)$ has already been set equal to

$q(\pi_m)$ for some slot m . Therefore we have

$$\sum_{s \in A} p(s) = \sum_{m \in A} q(\pi_m),$$

which, by additivity, is equal to $q(\theta)$, where $\theta = \sum_{m \in A} \pi_m$. We state this important result as our second corollary.

Corollary 2 *Every no-arbitrage lottery price function has a corresponding sunspot price function.*

5 Comparing Equilibrium Allocations

We are now in a position to address our main question: Under what conditions do the two models generate the same sets of equilibrium allocations? We begin by showing that this occurs whenever the prices supporting an allocation as an equilibrium in one model can be translated into the other model. We then provide two examples to illustrate the types of situations in which equivalence can fail.

5.1 Conditions for Equivalence

Corollary 2 states that any no-arbitrage lottery price function has a corresponding sunspot price function. We begin by restricting sunspot prices to be such that they have corresponding lottery prices. We show that, under this restriction, the two models generate the same set of equilibrium allocations. Any difference in these sets must therefore result from the extra flexibility of the sunspot pricing function. Using Proposition 3, we can rewrite the consumer's lottery-choice problem (3) as

$$\begin{aligned} & \max_{\delta_h, c_{0h}} \sum_{k=1}^K \delta_h(c_k) u_h(c_k) + v_h(c_{0h}), \\ & \text{subject to } \sum_{k=1}^K q(\delta(c_k)) \cdot c_k + c_{0h} \leq q(1) \cdot e_h + e_{0h}, \\ & \delta_h \in \Gamma(C), \quad c_{0h} \in \mathbb{R}_+. \end{aligned} \tag{13}$$

We now present the (restricted) equivalence result. Let (x^*, x_0^*) be a sunspot allocation and let (δ^*, c_0^*, y^*) be the corresponding lottery allocation generated by equations (7), (8), and (9). Let Q denote the set of all functions $q : \Gamma \rightarrow \mathbb{R}_+^L$, and recall that \widehat{P} is the set of sunspot price functions

satisfying (10). We then have the following.

Proposition 5 *There exists a $p^* \in \widehat{P}$ such that (p^*, x^*, x_0^*) is a sunspot equilibrium if and only if there exists a $q^* \in Q$ such that $(q^*, \delta^*, c_0^*, y^*)$ is a lottery equilibrium.*

This result establishes that when the price systems in the two models are comparable, the equilibrium allocations are identical and the differences in the trading stories do not matter. Combined with Corollary 2, it implies that every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation. This provides a partial answer to our main question.

Corollary 3 *Every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation.*

Under what conditions do sunspot equilibrium prices necessarily have corresponding lottery prices, so that full equivalence between the sets of equilibrium allocations obtains? GKQS [6] show that full equivalence arises when the sunspot variable is continuous and lottery choice is unconstrained. We now present a result that gives a sufficient condition for equivalence to hold when randomization possibilities are finite. First, Proposition 2 implies that when all slots on the wheel are equally likely, equilibrium prices in the sunspots model are linear in probabilities (Corollary 1 states that the same is true for the lottery model). Hence in this case the sets of equilibrium allocations coincide.

Corollary 4 *When the randomizing device is such that all slots on the wheel are equally likely, the set of sunspot equilibrium allocations is equivalent to the set of lottery equilibrium allocations.*

How likely is it that equivalence will fail? For many randomizing devices, condition (10) holds by default for all sunspot price functions. Suppose that there do not exist disjoint subsets A and B of S such that

$$\sum_{s \in A} \pi(s) = \sum_{s \in B} \pi(s) \quad (14)$$

holds. For example, when there are two states of nature (14) cannot hold if the states are not equally likely. In such cases, (10) imposes no restriction at all on the form of the sunspot price function. In other words, the sets P and \widehat{P} are the same, and therefore the sets of sunspot and lottery equilibrium allocations must be the same. As our final proposition, we show that the set of finite randomizing devices for which (10) actually imposes some restriction, and hence might

possibly be violated, is small – it has Lebesgue measure zero. Hence the equivalence of sunspot and lottery equilibrium allocations obtains for “almost all” finite randomizing devices.

Proposition 6 *Given any fundamental economy and any number M of slots, the set of randomizing devices for which the equivalence of the sets of sunspot and lottery equilibrium allocations fails has Lebesgue measure zero.*

To see why this proposition is true, consider the set of all randomizing devices with M slots; this set is equivalent to the $(M - 1)$ -dimensional unit simplex. Each of these devices naturally generates a sunspot variable with M states of nature. We want to ask which of these sunspot variables have the property that there exist nonempty, disjoint subsets A and B of S satisfying (14). The answer is that very few of them do; for almost every vector of M probabilities, (14) does not hold for any subsets A and B . Suppose, for example, that there are three states of nature. Then one way of creating disjoint subsets of S is to let A be the first state and B the second. In this case, (14) holds if $\pi(1) = \pi(2)$ holds. This condition defines a line segment in the two-dimensional unit simplex; hence, the set of three-state sunspot variables for which $\pi(1) = \pi(2)$ holds has Lebesgue measure zero. There are five other ways of defining sets A and B , and for each of these ways (14) also holds on a set of Lebesgue measure zero. The total set of randomizing devices for which (14) holds for some A and B is therefore a finite union of sets of measure zero, and hence has measure zero itself. When the wheel has more than three slots, the argument is the same except that the number of ways of defining A and B is larger. The details are in the appendix. The bottom line of this reasoning is that for a “typical” finite randomizing device, the sets P and \hat{P} are equal and Proposition 5 therefore establishes full equivalence of the sets of equilibrium allocations.

Note that the argument above gives a sufficient condition for equivalence; it does *not* say that whenever we have disjoint sets A and B satisfying (14) we will necessarily have nonequivalence. Equivalence will also obtain with many other randomizing devices. For example, when there are four states of nature, with $\pi(1) = \pi(2) = 1/10$ and $\pi(3) = \pi(4) = 2/5$, there are several ways of creating sets A and B such that (14) holds. Nevertheless, as we mentioned in Section 4.1, applying Proposition 2 (twice) shows that equivalence necessarily obtains under this device. Furthermore, a randomizing device where all slots are equally-likely satisfies (14), but Corollary 4 shows that equivalence always obtains with such device. What Proposition 6 shows is that the set of devices for which equivalence fails to obtain is a subset of a set of measure zero, and therefore has measure zero itself.

This result has an obvious probabilistic interpretation. Suppose the randomizing device were chosen at random in the following way. First M is drawn from any distribution over the set of integers greater than one. Then a randomizing device is drawn from the set of M -slot devices using any distribution that has a density with respect to Lebesgue measure. With probability one the chosen randomizing device will then be such that (10) places no restriction on the form of the sunspots price function, and therefore equivalence between the sets of sunspot and lottery equilibrium allocations obtains. However, we offer no theory of how the randomizing device is selected; it is certainly not clear that it should be viewed as being randomly selected. A government regulating risk classes in insurance, for example, might impose a device with “critical” properties in order to achieve a desired allocation. Hence we cannot dismiss sets of Lebesgue measure zero as being unimportant.

5.2 Examples of Nonequivalence

We now provide a pair of examples of sunspot equilibrium allocations with no lottery equilibrium counterpart. Our approach is to construct an allocation so that the supporting price vector necessarily violates condition (10). It is then immediate from Proposition 5 that the corresponding lottery allocation cannot be part of an equilibrium of the lottery model.

Example 1: There are three states of nature, with

$$\pi(1) = \frac{1}{2} \text{ and } \pi(2) = \pi(3) = \frac{1}{4}. \quad (15)$$

There is a single indivisible good, which must be consumed either in one unit or not at all. There are three consumers, all of whom have preferences given by $u_h(0) = 0$, $u_h(1) = 1$, and $v_h(x_{0h}) = x_{0h}/100$.¹⁴ The endowments of the indivisible good are

$$(e_1, e_2, e_3) = \left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10} \right),$$

so that the aggregate endowment of this good is 1. All consumers are endowed with zero units of the divisible good. We have designed this example so that at most prices, consumers will want to buy as much of the indivisible good as they can. The sole role of the divisible good to provide consumers with a productive use for any income that is not spent on indivisible goods. With the

¹⁴ The linearity of the function v_h is not important; all that matters for this example is that the marginal utility of consumption of the divisible good at zero be small enough.

price of the divisible good normalized to unity, let the prices of the indivisible good in each state be given by

$$p^* = (p^*(1), p^*(2), p^*(3)) = \left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10} \right).$$

and let the allocation of the indivisible good be

$$x_1^* = (1, 0, 0), \quad x_2^* = (0, 1, 0), \quad \text{and} \quad x_3^* = (0, 0, 1).$$

Of course, everyone consumes zero of the divisible good. It is straightforward to verify that this pairing of prices and allocations is a sunspot equilibrium. It can also be shown that the equilibrium allocation is unique up to a relabelling of states 2 and 3. In other words, the only equilibrium allocations for this economy are x^* and the allocation that is equal to x^* but with the bundles given to consumers 2 and 3 reversed.

It is clear that the price vector p^* violates (10), because consuming the good with probability one-half costs either $2/5$ (if state 1 is chosen) or $3/5$ (if states 2 and 3 are chosen). However, showing that the allocation x^* has no lottery equilibrium counterpart requires showing something stronger: that *no* price vector satisfying (10) supports x^* as a sunspot equilibrium. In strictly convex economies with smooth preferences, the price vector supporting a given equilibrium allocation is necessarily unique (up to the normalization) because it is determined by the tangency of the indifference surfaces at the given allocation. With nonconvexities or with kinks in the indifference surfaces, however, it is possible for many price systems to support the same allocation as an equilibrium, even from the same endowment point. In this example, there are indeed many price vectors that support x^* as an equilibrium, but all are a scaling up or down of the vector p^* . In other words, the equilibrium value of the divisible good is not uniquely determined, but the relative values of the indivisible good in each state are. This implies that any prices supporting x^* as an equilibrium must violate condition (10), and therefore by Proposition 5 the lottery allocation corresponding to x^* cannot be part of an equilibrium of the lottery economy. \square

This example is similar in spirit to one given in Shell and Wright [16, pp. 9-10]. If the sunspot variable were continuous, in equilibrium each consumer would receive the indivisible good with a probability equal to her endowment. In an equilibrium based on the finite device generating (15), however, consumption in state 1 trades at a discount and consumption in states 2 and 3 trade

at a premium relative to their respective probabilities. As a result, the first consumer is able to consume the divisible good with a probability that is higher than her endowment, while the other two consumers receive probabilities lower than their endowments. These latter consumers are willing to pay the premium because they cannot afford to consume the indivisible good in the (relatively) discounted state – the state is too “large” and hence too expensive in absolute terms. The only other option they have is to consume the divisible good, which gives very little utility. Thus this particular randomizing device generates a “volume discount” for probability. Notice that the same outcome could be achieved with a continuous sunspot variable if individual endowments were changed to match the probability structure in (15). Hence, in the context of this example, regulating stochastic trade can substitute for lump-sum taxes and transfers. This type of result does not depend on the exact specification of probabilities in (15). If, for example, we were to change the probabilities slightly, to

$$\pi(1) = 0.52 \text{ and } \pi(2) = \pi(3) = 0.24, \quad (16)$$

(p^*, x^*) would still be a sunspot equilibrium. The equilibrium prices in this example are determined by the endowments; small changes in the probabilities will only affect the size of the volume discount (or, the amount of redistribution).

The pattern of costs represented by p^* can be replicated in the lottery model by a price function q . However, under the randomizing device represented by (15) this function does not satisfy additivity. In particular, because there are two slots on the wheel with probability one-quarter, additivity requires that the cost of consuming the indivisible good with probability one-half be equal to twice the cost of consuming it with probability one-quarter. Therefore, the lottery prices capable of replicating the cost structure contained in p^* would generate an arbitrage opportunity for the lottery-producing firm, and hence clearly cannot be equilibrium prices. In fact, it is straightforward to verify that no lottery equilibrium exists for this example. Suppose, however, that the randomizing device is changed slightly, say, to that represented in (16). Then condition (10) is satisfied (by default) and the sunspot equilibrium allocation discussed above *does* have a lottery equilibrium counterpart. This fact is in line with Proposition 6: for almost all specifications of the probabilities of the three states, the sets of equilibrium allocations are equal. However, for a set of “critical” devices, like that represented in (15), the lottery model may be incapable of replicating the sunspot equilibrium prices, which leads to the nonequivalence demonstrated in the example above.

One might be tempted to think that this example of nonequivalence, which relies on the additivity restriction for lottery prices, is an artifact of our particular definition of the constrained lottery model. Perhaps if the mechanism for producing lotteries had been defined differently, additivity of prices need not be imposed and full equivalence between the sets of equilibrium allocations would obtain. We now present a slightly richer example to show that this is not the case. In this second example, the sunspot equilibrium will have two different consumers purchasing the same consumption bundle with the same probability, but paying a different price for it because they purchase it in different states. This is something that simply cannot happen in the lottery model, because in the lottery model probabilities (rather than states) define the basic commodities to which the law of one price applies.

Example 2: There are three states of nature, with probabilities given by

$$\pi(1) = 0.5, \pi(2) = 0.4, \text{ and } \pi(3) = 0.1.$$

Notice that, as in the first example, there are two disjoint subsets of states that have probability one-half: state 1 alone and states 2 and 3 together. There are three consumers, all of whom have the consumption set $\mathcal{C} = C_h = \{0, 1, 2\}$, and have utility functions that are linear in consumption: $u_h(c_k) = c_k$ for all h and k .¹⁵ The total endowment in each state is 3 units of the good, which is divided into the following private endowments

$$(e_1, e_2, e_3) = (0.65, 0.45, 1.90).$$

We normalize prices so that $\sum_s p(s) = 1$ holds. Suppose that the price vector is given by

$$p^* = (p^*(1), p^*(2), p^*(3)) = (0.55, 0.40, 0.05).$$

Then consumption in state 1 trades at a premium and consumption in state 3 trades at a discount relative to their respective probabilities. At these prices, the consumption bundles demanded by

¹⁵ There is no divisible good in this example. However, in the equilibrium we construct all consumers exhaust their income. Because of this, it is straightforward to add a divisible good with a zero total endowment, as in the first example, and keep the equilibrium allocation (and price) of the indivisible good the same. We go through the example without the divisible good to simplify the presentation; we discuss below how to add the divisible good. Similarly, the assumption of risk neutrality is not important. Each consumer will be choosing a strictly most-preferred bundle from a finite set, and therefore small perturbations of the individual utility functions will have no effect on the equilibrium.

each consumer are given by

$$x_1^* = (1, 0, 2), \quad x_2^* = (0, 1, 1), \quad \text{and} \quad x_3^* = (2, 2, 0).$$

These demands can be verified by computing the cost and the utility level associated with each of the $3^3 = 27$ possible consumption bundles. Because total demand for the good is equal to the total supply in each state, (p^*, x^*) is a sunspot equilibrium.

It is clear that the price vector p^* violates (10), but once again we must show that no price vector satisfying (10) supports x^* as a sunspot equilibrium. To see that this is indeed the case, first note that the fact that no resources are wasted in equilibrium implies that every consumer must exhaust her income in equilibrium.¹⁶ Therefore, at any prices that support x^* as an equilibrium, the three budget constraints must hold with equality, as must our price normalization. The budget constraints are not independent equations – an allocation that does not waste resources and satisfies the first two will also satisfy the third at any prices. Therefore we drop the third budget constraint and, writing the individual consumption plans x_h^* as row vectors and using $\mathbf{1}$ to denote a vector of ones, arrive at the system of equations

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ 1 \end{bmatrix}.$$

There are no prices on the right-hand side of the budget constraints because of our choice of price normalization. It is easy to verify that the left-most matrix is full rank. As a result, there is a unique solution of this equation, which is given by the equilibrium prices p^* . In other words, there is only one normalized price vector that makes the consumption plan x_h^* affordable from endowment e_h for all three consumers. Thus there clearly cannot exist another price vector that supports x^* as a sunspot equilibrium. Because the unique supporting price vector p^* violates (10), using Proposition 5 we can conclude that the corresponding lottery cannot be an equilibrium allocation of the lottery economy. \square

In this sunspot equilibrium, consumer 1 receives one unit of the good in state 1 and pays $p(1) = 0.55$ for it. At the same time, consumer 2 receives one unit of the good in states 2 and 3 at a cost of $p(2) + p(3) = 0.45$. In other words, two consumers are buying the same consumption bundle

¹⁶ If some consumer were not spending all of her income, the value of total demand would be less than the value of total supply. This would imply that aggregate consumption is less than the aggregate endowment in some state, which is clearly not the case here.

with the same probability and paying *different* prices for it. This is something that simply cannot happen in the lottery model. Why is consumer 1 willing to buy one unit of the good in the more expensive state (state 1) rather than switching to the cheaper states (2 and 3)? The reason is that buying one unit in state 1 allows her to be able to purchase two units of the good in state 3. If she purchased one unit of the good in states 2 and 3 instead of in state 1, she would then be unable to consume any of the good in state 1, leaving her strictly worse off. In other words, consumption in states 2 and 3 is not a perfect substitute for consumption in state 1, because consuming in states 2 and 3 restricts the available options for consuming in the remaining one-half of probability.

It is straightforward to add a divisible good to this example and verify that the result does not depend on the absence of local nonsatiation. As in Example 1, endow all consumers with zero units of the divisible good and set the price of this good to unity. Then by choosing the functions v_h so that the marginal utility of consumption of the divisible good at zero is small enough, we can guarantee that each consumer will demand zero units of the divisible good and the same consumption plan for the indivisible goods. (For this example, a marginal utility of one-half for all consumers will work.) Therefore we have an equilibrium of the expanded economy with the same prices and allocation of the indivisible good as given above.¹⁷ To clarify the intuition for why this trick works, consider again the question of why consumer 1 is willing to buy one unit of the good in the expensive state. If she were to instead buy one unit in states 2 and 3, she would again be unable to afford any of the indivisible good in state 1. However, she can now spend her leftover income on the divisible good. With the price of the divisible good set to unity, she can afford 0.2 units. The trick is to set her utility function v_h so that she would rather have 2 units of the indivisible good with probability 0.1 than these 0.2 units of the divisible good with certainty. Then she is still willing to pay the premium to consume in state 1, and our equilibrium allocation is unchanged.

6 Conclusions

In this paper, we have extended the analysis of the relationship between sunspot equilibrium and lottery equilibrium allocations to a class of completely finite models. Previous work based on

¹⁷ The supporting price vector for the equilibrium allocation is no longer unique; a range of prices for the divisible good will lead to the same outcome. However, all supporting price vectors will have the same relative prices for the indivisible goods in different states, and hence will violate (10).

exchange economies with non-convexities has shown that when the randomizing device is continuous, the two sets of equilibrium allocations are equivalent. Here we have focused on the case where the randomizing device is discrete and the number of possible lotteries is finite. We have modified the lottery model so that it can be applied to such economies. Our main finding is that equivalence between the two sets of equilibrium allocations often, but not always, obtains.

The key difference between the two models, and the source of potential nonequivalence, is in their respective price systems. We show that equivalence will hold unless prices in the sunspots model are such that buying a particular consumption bundle with a particular probability has two (or more) different costs assigned to it, depending on which states of nature the bundle is purchased in. For all randomizing devices except a set of Lebesgue measure zero, this cannot happen simply because each level of probability is generated by a unique combination of states. A separate argument shows that equivalence also obtains for the “leading” case where all events are equally likely. In this case, it is the ability of consumers to substitute, or “shift,” their purchases from high-cost to low-cost states that is critical for the result. Hence the equivalence result proven in GKQS [6] does not depend on having a continuous sunspot variable (or, unconstrained randomization possibilities). It naturally extends to large class of models with finite randomization possibilities.

We also present two examples of nonequivalence: sunspot equilibrium allocations with no lottery equilibrium counterpart. In these examples, there are different ways of combining states together to arrive at the same total probability. In such cases, the extra generality available in the price system in the sunspots model can be important. This result is similar in spirit to that of Garratt [4], who showed that all lottery equilibrium allocations have corresponding sunspot equilibrium allocations, but that the converse is not true. It is important to bear in mind, however, that the results here are fundamentally different because we are using a generalized lottery model that constrains the randomization opportunities available to agents. The sunspot equilibrium allocation that Garratt [4] shows to have no lottery equilibrium counterpart *does* have a counterpart in our constrained lottery model. Our example of nonequivalence based on the additional flexibility of prices in the sunspots model is an entirely new and different phenomenon. We have redefined the lottery model to bring it as close as possible to the sunspots model. Nonetheless, there are still some sunspot equilibrium allocations that are not lottery equilibrium allocations.

As a final note, we reiterate that the environment we have studied in this analysis is special. Markets are perfect, the number of commodities in the underlying certainty economy is finite,

there is no role for money, etc. Sunspot equilibrium has been applied in a much wider range of settings. It is unclear whether and how the lottery model can be extended to many of these environments. A recent step in this direction was made by Berentsen, Molico, and Wright [2], who introduced lotteries into search-theoretic models of money. Whenever the lottery model is extended to a new environment, it is natural to ask whether or not equivalence of the two sets of equilibrium allocations obtains. We expect that the basic approach we have taken here, if not our specific results, will be useful in addressing the equivalence question whenever it arises.

Appendix A. Proofs of Propositions

Proof of Proposition 1: (a) : Suppose that (x^*, x_0^*) is feasible, so that we have $x^*(s) \in F$ for all s and $\sum_{h \in H} x_{0h} \leq \sum_{h \in H} e_{0h}$. To show that the corresponding lottery allocation is also feasible, we need to show that (i) $\sum_{h \in H} c_{0h} \leq \sum_{h \in H} e_{0h}$ holds and (ii) $y \in Y$ holds. (That the individual lotteries being consumed are equal to those being produced is guaranteed by the definition of y in (9)). The first of these conditions is immediate from the definition of c_{0h} in (8).

For the second, we need to construct the assignment functions that distribute each individual lottery on the roulette wheel. From (9), we see that the firm is buying one degenerate lottery from each consumer (her endowment) and selling one possibly-nondegenerate lottery to each consumer (her consumption). Arranging the degenerate lotteries is trivial (the same bundle is assigned to every slot). For the possibly-nondegenerate lotteries, let $f : M \rightarrow S$ be the function that maps slots on the wheel to states in the sunspots model. We then define the assignment function for individual lottery δ_h by

$$g_h(m) = x_h(f(m)) \quad \text{for all } m \in M. \quad (17)$$

In other words, the lottery-producing firm arranges the stochastic allocations on the wheel in exactly the same way that they are arranged in the sunspots model. Then (4) must clearly hold for each h . Furthermore, we can write (5) as follows

$$\begin{aligned} \sum_{j,n} I_{j,n} g_{j,n}(m) &= \sum_{h \in H} (-e_h + g_h(m)) \\ &= \sum_{h \in H} (-e_h + x_h(f(m))) \\ &= \sum_{h \in H} (-e_h + x_h(s)) \leq 0 \quad \text{for all } s. \end{aligned}$$

In this way, feasibility of the sunspot allocation guarantees feasibility of the lottery allocation.

(b) : The proof of the converse is essentially the same argument in reverse. We start with a feasible lottery allocation and a sunspot allocation that induces it. Feasibility of the lottery allocation implies the existence of assignment functions g_h , which correspond to the sunspot consumption plans as in (17) above. The lottery feasibility condition (5) then implies that we have $x(s) \in F$ for

all s , and therefore the sunspot allocation is also feasible. ■

Proof of Proposition 2: We prove this proposition in steps, beginning with a pair of lemmas.

Lemma 1 *For any equi-probable set A , if x_h satisfies*

$$\sum_{s \in S} p(s) \cdot x_h(s) \leq \sum_{s \in S} p(s) \cdot x_h^t(s)$$

for $t = 1, \dots, N_A$, then we have

$$\sum_{s \in S} p(s) \cdot x_h(s) \leq \sum_{s \in S} \bar{p}_A(s) \cdot x_h(s).$$

Proof of Lemma 1: The hypothesis of the lemma implies that we have

$$\begin{aligned} \sum_{s \in S} p(s) \cdot x_h(s) &\leq \frac{1}{N_A} \sum_{t=1}^{N_A} \sum_{s \in S} p(s) \cdot x_h^t(s) \\ &= \sum_{s \in S} p(s) \cdot \sum_{t=1}^{N_A} \frac{x_h^t(s)}{N_A}. \end{aligned}$$

The right-hand side of this inequality replaces consumption in state s with the average consumption over all states in A . Because this average is the same for all s in A , we can replace the price vector p with \bar{p}_A without changing the value of the inner product;

$$\sum_{s \in S} p(s) \cdot \sum_{t=1}^{N_A} \frac{x_h^t(s)}{N_A} = \sum_{s \in S} \bar{p}_A(s) \cdot \sum_{t=1}^{N_A} \frac{x_h^t(s)}{N_A}.$$

Note that since \bar{p}_A takes on the same values for all s in A , we can “undo” the averaging of the allocation. In other words, for states in A , it makes no difference if we multiply the average price by the average consumption in each state or if we multiply the average price by the actual consumption in each state;

$$\sum_{s \in S} \bar{p}_A(s) \cdot \sum_{t=1}^{N_A} \frac{x_h^t(s)}{N_A} = \sum_{s \in S} \bar{p}_A(s) \cdot x_h(s).$$

This establishes the desired result. □

The next lemma shows that we can make a stronger statement about equilibrium prices.

Lemma 2 Suppose (p^*, x^*) is a sunspot equilibrium. Then for any equi-probable set A ,

$$\sum_{s \in S} p^*(s) \cdot x_h(s) = \sum_{s \in S} \bar{p}_A^*(s) \cdot x_h(s)$$

must hold for all h .

Proof of Lemma 2: Given local nonsatiation, individual optimization implies that an equilibrium allocation x_h^* must be the minimal cost element of any A -shift class $T_A(x_h^*)$. Therefore, by Lemma 1, we have

$$\sum_{s \in S} p^*(s) \cdot x_h^*(s) \leq \sum_{s \in S} \bar{p}_A^*(s) \cdot x_h^*(s)$$

for all h . Suppose that this holds with strict inequality for some h . Then summing this inequality across all consumers and using the fact that each consumer's budget constraint must hold with equality (again due to local nonsatiation) yields

$$\sum_{s \in S} p^*(s) \cdot \sum_{h \in H} e_h < \sum_{s \in S} \bar{p}_A^*(s) \cdot \sum_{h \in H} x_h^*(s).$$

But market clearing (or the feasibility of x^*) requires

$$\sum_{h \in H} x_h^*(s) \leq \sum_{h \in H} e_h$$

for every state s , we have

$$\sum_{s \in S} p^*(s) \cdot \sum_{h \in H} e_h < \sum_{s \in S} \bar{p}_A^*(s) \cdot \sum_{h \in H} e_h.$$

Because we have $\sum_{s \in S} p^*(s) = \sum_{s \in S} \bar{p}_A^*(s)$, the above inequality is a contradiction. \square

With these two lemmas in hand, we are ready to prove the statement in the proposition. It suffices to show that (x_h^*, x_{0h}^*) is still an optimal choice for consumer h when prices are given by \bar{p}_A^* . Lemma 2 shows that (x_h^*, x_{0h}^*) is still affordable at these prices. Suppose it is not optimal for some consumer h . Then there exists some other plan $(\tilde{x}_h, \tilde{x}_{0h})$ that is affordable at prices \bar{p}_A^* and is strictly preferred to (x_h^*, x_{0h}^*) . Thus we would have

$$\sum_{s \in S} \bar{p}_A^*(s) \cdot \tilde{x}_h(s) + \tilde{x}_{0h} \leq \sum_{s \in S} \bar{p}_A^*(s) \cdot e_h + e_{0h} = \sum_{s \in S} p^*(s) \cdot e_h + e_{0h}.$$

Let \tilde{y}_h denote the minimum cost element of $T_A(\tilde{x}_h)$ at prices p^* . Then $(\tilde{y}_h, \tilde{x}_{0h})$ is also strictly preferred to (x_h^*, x_{0h}^*) and \tilde{y}_h costs exactly the same as \tilde{x}_h at prices \bar{p}_A^* . Therefore we have

$$\sum_{s \in S} \bar{p}_A^*(s) \cdot \tilde{y}_h(s) + \tilde{x}_{0h} \leq \sum_{s \in S} p^*(s) \cdot e_h + e_{0h}.$$

Because \tilde{y}_h is the minimum cost element of its A -shift class, Lemma 1 implies that we have

$$\sum_{s \in S} p^*(s) \cdot \tilde{y}_h(s) \leq \sum_{s \in S} \bar{p}_A^*(s) \cdot \tilde{y}_h(s),$$

meaning that $(\tilde{y}_h, \tilde{x}_{0h})$ was affordable at prices p^* . This contradicts the optimality of x_h^* at prices p^* . ■

Proof of Proposition 3: Let $e^\ell \in C$ denote the commodity bundle that has one unit of good ℓ and zero units of every other good, and let $\mathbf{0} \in C$ denote the zero vector; i.e., the commodity bundle that contains zero units of every indivisible good. Let $\delta^{(\theta, \ell)}$ denote the lottery that delivers the commodity bundle e^ℓ with probability θ and the commodity bundle $\mathbf{0}$ with probability $(1 - \theta)$. Note that this lottery is in $\Gamma(C)$ if and only if θ is in Γ . Define the function $q : \Gamma \rightarrow \mathbb{R}_+^L$ by

$$q_\ell(\theta) \equiv \phi\left(\delta^{(\theta, \ell)}\right)$$

for every $\theta \in \Gamma$ and for $\ell = 1, \dots, L$. Consider an arbitrary lottery $\hat{\delta} \in \Gamma(C)$ and suppose that

$$\phi(\hat{\delta}) > \sum_{k=1}^K q(\hat{\delta}(c_k)) \cdot c_k \tag{18}$$

held. (The reverse case is completely symmetric.) We will show that the firm could then make unbounded profits by purchasing the components needed to construct $\hat{\delta}$ separately and selling the lottery $\hat{\delta}$. Let c_k^ℓ denote the number of units of good ℓ contained in the k^{th} commodity bundle, and define $\theta_k \equiv \hat{\delta}(c_k)$. Consider the following production plan. For all k and ℓ , set

$$y\left(\delta^{(\theta_k, \ell)}\right) = c_k^\ell.$$

Then the firm is buying c_k^ℓ units of the lottery that delivers one unit of good ℓ with the same probability that the lottery $\hat{\delta}$ assigns to the commodity bundle c_k . The total cost of purchasing these lotteries is equal to the right-hand side of (18). Set $y(\hat{\delta}) = 1$, so that the firm is selling

one unit of the lottery $\widehat{\delta}$. Set all other values of y equal to zero. Under (18), this plan generates positive profits. All that remains is to show that this plan is feasible, i.e., that there exist assignment functions such that (4) and (5) are satisfied.

Because $\widehat{\delta}$ is in the set $\Gamma(C)$, we know that there exists a function \widehat{g} such that

$$\widehat{\delta}(c_k) = \sum_{m:\widehat{g}(m)=c_k} \pi_m$$

holds for all k . For each lottery $\delta^{(\theta_k, \ell)}$, let the arrangement function be denoted by $g_{k, \ell}$ and set

$$g_{k, \ell}(m) = \begin{cases} e^\ell & \text{if } \widehat{g}(m) = c_k \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

Condition (4) is then satisfied automatically: the arrangement $g_{k, \ell}$ delivers one unit of good ℓ with probability $\theta_k = \widehat{\delta}(c_k)$, which is the definition of the lottery $\delta^{(\theta_k, \ell)}$. Recall that the firm is purchasing c_k^ℓ units of this lottery; arrange all of these units according to the same function $g_{k, \ell}$. Let g_k denote the vector of functions $(g_{k, 1}, \dots, g_{k, L})$. Then the vector of net resources used by the firm if slot m is realized is given by

$$\widehat{g}(m) - \sum_{k=1}^K I_{(\widehat{g}(m)=c_k)} g_k(m) \cdot c_k = 0$$

for all m , where I is the standard indicator function. Hence this production plan satisfies (5) and is feasible. Since this plan can be replicated at an arbitrarily large scale, under any price system satisfying (18) (or satisfying the reverse inequality) the firm's problem (6) has no solution. ■

Proof of Proposition 4: Suppose prices are not additive. Using Proposition 3, this would imply that there exists a set $A \subseteq M$ of slots such that $q_\ell \left(\sum_{m \in A} \pi_m \right) \neq \sum_{m \in A} q_\ell(\pi_m)$ for at least one good ℓ . Suppose that

$$q_\ell \left(\sum_{m \in A} \pi_m \right) > \sum_{m \in A} q_\ell(\pi_m) \tag{19}$$

holds. (The reverse case is symmetric.) Consider the following production plan. For each slot $m \in A$, the firm buys one unit of the lottery that delivers one unit of good ℓ with probability π_m and nothing with probability $(1 - \pi_m)$. The firm sells one unit of the lottery that delivers one unit of good ℓ with probability $(\sum_{m \in A} \pi_m)$ and nothing with probability $(1 - \sum_{m \in A} \pi_m)$. This plan

is feasible by construction – the firm takes in and gives out one unit of good ℓ if one of the slots in A is realized, and does nothing otherwise. Under (19), this plan yields a strictly positive profit. Since the firm could replicate this plan on an arbitrarily large scale, problem (6) has no solution. ■

Proof of Proposition 5: (a) : First, suppose that (p^*, x^*, x_0^*) is a sunspot equilibrium with $p^* \in \widehat{P}$. Then we know that at prices p^* , (x_h^*, x_{0h}^*) solves problem (2) for every h . By applying a change of variables to replace the summation across states with a summation across the consumption set, we can rewrite this problem as

$$\begin{aligned} & \max_{x_h} \sum_{k=1}^K u_h(c_k) \pi \circ x_h^{-1}(c_k) + v_h(x_{0h}), \\ \text{subject to } & \sum_{k=1}^K \left(\sum_{s \in x_h^{-1}(c_k)} p^*(s) \cdot c_k \right) + x_{0h} \leq \sum_{s \in S} p^*(s) \cdot e_h + e_{0h}, \\ & x_h \in X, x_{0h} \in \mathbb{R}_+. \end{aligned}$$

Here, $x_h^{-1}(c_k)$ is the set of states in which the consumer buys the bundle c_k . Since $p^* \in \widehat{P}$ holds, we can use (11) together with the definitions $\delta_h^* = \pi \circ x_h^{*-1}$ and $c_{0h} = x_{0h}$ to write the problem as

$$\begin{aligned} & \max_{\delta_h} \sum_{k=1}^K \delta_h(c_k) u_h(c_k) + v_h(c_{0h}), \\ \text{subject to } & \sum_{k=1}^K (q^*(\delta_h(c_k)) \cdot c_k) \leq q^*(1) \cdot e_h, \\ & \delta_h \in \Gamma(C), c_{0h} \in \mathbb{R}_+, \end{aligned}$$

which is exactly the consumer's lottery problem as given in (13). Hence there exists a $q^* \in Q$ at which (δ_h^*, c_{0h}^*) solves problem (13) for all h , and condition (i') in the definition of lottery equilibrium is satisfied.

Condition (ii') requires that y^* be an optimal choice for the firm at prices q^* . It is easy to verify that y^* yields zero profits at these prices. Because the prices are of the no-arbitrage form (see Section 4.2), positive profits are not possible and therefore the firm is indeed optimizing. Condition (iii') follows directly from Proposition 1. Therefore we have a $q^* \in Q$ such that $(q^*, \delta^*, c_{0h}^*, y^*)$ is a lottery equilibrium.

(b) : Now suppose that $(q^*, \delta^*, c_{0h}^*, y^*)$ is a lottery equilibrium with $q^* \in Q$. Because equilibrium

prices must be of the no-arbitrage form, reversing the argument above shows that at the unique price function p^* corresponding to q^* through (11), (x_h^*, x_{0h}^*) is optimal for each consumer h . Therefore condition (i) in the definition of a sunspot equilibrium is satisfied. Note that this p^* is in the set \widehat{P} by definition. Condition (ii) in this definition follows directly from Proposition 1. Therefore we have a $p^* \in \widehat{P}$ such that (p^*, x^*, x_0^*) is a sunspot equilibrium. ■

Proof of Proposition 6: The set of all M -slot randomizing devices can be represented as the $(M - 1)$ -dimensional unit simplex, which we denote by Σ_M . The possibility that there exists a sunspot price function with no lottery counterpart can only arise if there exist nonempty, disjoint sets $A, B \subset M$ such that (14) holds. If there are no such sets A and B , then the sets \widehat{P} and P are identical and Proposition 5 establishes full equivalence between the sets of equilibrium allocations. Let α_n denote the number of ways of choosing exactly n of the M slots. Let β_n be the number of ways of dividing these n elements into two nonempty, disjoint groups. For any finite M and n , both α_n and β_n are finite numbers. There are then

$$\sum_{n=2}^M \alpha_n \beta_n$$

ways of creating nonempty, disjoint sets A and B that might satisfy (14); this number is also finite. For each possible (A, B) combination, (14) holds on a linear $(M - 2)$ -dimensional subset of Σ_M , which has Lebesgue measure zero in Σ_M . Hence the set of randomizing devices satisfying (14) is a finite union of sets of Lebesgue measure zero, and therefore itself has Lebesgue measure zero in Σ_M . Since we know that equivalence of the sets of equilibrium allocations obtains for all randomizing devices except possibly those for which (14) holds, we have established the proposition. ■

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